The Time Cost of Information in Financial Markets

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Abstract

I model a financial market in which traders acquire private information through time-consuming research. A time cost of information arises due to competition - through the expected adverse price movements due to others' trades - causing traders to rush to trade on weak information. This cost monotonically increases with asset value uncertainty, so that, exactly opposite to the result under the standard modeling assumption of a monetary cost of information, traders acquire the least information when this uncertainty is largest. The model makes several novel testable predictions regarding volume and order imbalances, some of which have existing empirical support.

1 Introduction

Incentives to acquire information about financial assets are crucial to the informational efficiency of market prices, which is in turn important for the efficient allocation of resources, specifically capital.[1] As such, a large literature studies information acquisition in financial

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[1]The idea that prices serve an important role in allocating resources goes back to at least Hayek (1945). In particular, firm prices are socially valuable because they allow capital to be allocated efficiently across firms and serve as a signal to managers that internal resources are being used appropriately. See Bond, Edmans, and Goldstein (2012) for a recent review of the literature on the real effects of secondary financial markets.

market settings (examples include Grossman and Stiglitz (1980), Verrecchia (1982), Admati and Pfleiderer (1988), Kyle (1989), Barlevy and Veronesi (2000,2007), Veldkamp (2006a,b), Chamley (2007), Glode, Green, and Lowery (2012), Lew (2013), Manela (2014), Nikandrova (2014), Biais, Foucault, and Moinas(2015), and Banerjee and Breon-Drish (2017)). This literature typically models the cost of information in monetary (or utility) units - as some weakly increasing function of signal quality. This monetary cost may be interpreted literally, or, given that it takes time to acquire and process information into a trading decision, as the reduced-form of a time cost. In this paper, I model the time cost explicitly, showing that it takes a very different form than is typically assumed and that this difference has important implications for the informational efficiency of prices.

As in the existing literature, traders in the model can produce a private, informative signal about an asset's value. However, the quality of the signal is an increasing function of *time*, rather than of money.[2] In deciding how much research to do, a trader faces a trade-off: better information increases her expected profits through improved trading decisions. But, any information that becomes public (whether directly, via news, or indirectly, via inference from trades) moves prices closer to the asset's true value in expectation, eating into her expected profit.[3] The cost of research is therefore a function of more than just the signal quality a trader obtains. In fact, it varies in such a way that the conditions under which traders obtain better quality information are completely opposite to those in a model with a purely monetary cost.

The model is a two-period version of the classic trading model of Glosten and Milgrom (1985) in which risk-neutral agents trade an asset with a market maker.[4] I introduce an information acquisition decision in which a trader can obtain a relatively weak, private signal in the first period (which I call *rushing*) or, by spending time doing research, a stronger, private signal in the second period (which I call *waiting*). I first consider a single trader in order to illustrate the main force in the model in the simplest possible setting.[5] To

[2]More generally, one would expect signal quality to be a function of both time and money. In my analysis, I consider the two types of costs independently in order to emphasize the difference in equilibrium information acquisition strategies.

[3]In the model, prices reflect all available public information, so that public information imposes a cost. This contrasts with models of common investment opportunities, such as Chamley and Gale (1994), Gul and Lundholm (1995), and Chari and Kehoe (2004), in which information revealed by others' decisions may be beneficial.

[4]Related papers that allow for trade timing include Admati and Pfleiderer (1988), Foster and Viswanathan (1990), Smith (2000), Ostrovsky (2012), Bouvard and Lee (2015), Dugast and Foucault (2017), and Malinova and Park (2014). I discuss the most relevant of these papers in the following.

[5]It is a simple exercise to extend the model to allow sequential arrival as in the original Glosten and Milgrom (1985) model, provided that the information acquisition decisions of each trader do not interact (i.e. the subsequent trader arrives after the last possible time the previous trader trades). Extending the model in this manner, however, leads to relatively little new insight. Instead, I consider a second trader

introduce a cost of waiting in this setup, I allow arbitrary, public 'news' to arrive between the two periods.[6] In this setting, news 'crowds out' private information acquisition so that the overall informational efficiency of market prices depends upon which effect dominates.[7] In the main results, I show that competition between two traders similarly produces a cost of waiting so that each attempts to preempt the other. Intuitively, trades by others reveal information, so that the model of competition effectively endogenizes the exogenous public news arrival of the single trader setting. With competition, *both* traders rush more often than if in isolation, which (generally) lowers informational efficiency.

The key insight of the model is that public information has the largest impact on prices and profits, and therefore leads to the greatest time cost (in terms of forgone profits), precisely when private information is most valuable (in terms of the expected profits it generates) - when the variance in the asset value (conditional on all public information) is high. Standard intuition suggests that when uncertainty in the asset value (hereafter 'uncertainty') is high, information is most valuable because it creates a large difference between private and public beliefs. I confirm this intuition in a model with monetary costs by showing that traders are willing to pay the most for information when uncertainty is highest. However, because public information then imposes the largest time cost, I further show that traders actually do the *least* research at this time when we replace the monetary cost with either news or competition among traders. This difference in information acquisition strategies is consequential: at a time of high uncertainty, information contributes most to the long-term informational efficiency of the market, so that rather than acquiring more information when it most benefits market prices, the time cost of information causes traders to instead acquire less.

The key assumption for these results is that traders choose a signal quality prior to trading. This assumption is, for example, a natural consequence of a trader choosing where to look for information: reading the latest announcement of a firm is likely to be faster than calling an analyst, but is also likely to provide lower quality information. Because both activities consume time, if the trader reads the announcement, she cannot simultaneously be talking to the analyst. Under this interpretation, the prediction of the model is that fast, weak sources of information are more likely to be exploited when uncertainty is high and/or there is more competition. For example, firms specializing in stocks that are highly volatile and attract a lot of interest may prefer technologies that give them fast access to

whose information acquisition decision interacts with that of the first.

[6]The high-frequency trading models of Foucault, Hombert, and Roşu (2016) and Biais, Foucault, and Moinas(2015) also feature fast traders who have the opportunity to trade in front of public information. In each of these models, however, the fast traders get early access (either for free or at a monetary cost) to the public information itself, rather than having to decide how much private information to acquire.

[7]Similarly, firms' information disclosure decisions can crowd out private incentives to acquire information (Diamond (1985) and Boot and Thakor (2001)).

weak information. Firms that focus on more stable stocks may instead prefer slower, more in-depth research.

As in Glosten and Milgrom (1985), all trades take place with a risk-neutral market maker who faces perfect competition, earning zero expected profit. Traders can be either informed traders that choose the quality of their information, or uninformed traders that facilitate trade by trading for reasons exogenous to the model. The market maker accounts for the possibility that he faces an informed trader and posts separate bid and ask prices that are conditional on the order type. The difference between the bid and ask prices, the bid-ask spread, imposes a trading cost that the informed trader can attempt to avoid by masquerading as an uninformed trader. For example, if the informed trader were to always rush, the market maker would increase the first period spread accordingly. Then, however, the informed trader would prefer to wait, masquerading as an uninformed trader in the second period.

This strategic interaction naturally arises when a trader with market power decides when to trade, but is orthogonal to the main question of interest and is not necessary for the results.[8] In fact, in the main analytic results, I show that when the probability of informed trading is sufficiently small, the time cost of acquiring better information is the dominant force. Numerical simulations in the Online Appendix demonstrate that the analytic results continue to hold, except for probabilities of informed trading that exceed the largest estimates from empirical research. Furthermore, for large probabilities of informed trading, the simulations produce counterexamples which show that moderate levels of the probability of informed trading are not only sufficient, but also necessary for the main results.

The model makes several testable predictions about the effects of uncertainty, competition, and information quality on trade timing and order imbalances. Indirect evidence exists for some of these predictions, including that the order imbalance decreases with competition. Predictions with respect to uncertainty are more novel and do not yet have conclusive evidence from field data. Kendall (2018) tests a closely related model in a controlled laboratory experiment, providing evidence that competition induces rushed trades. There, I extend the model to more than two traders in order to study market rushes: episodes in which all traders rush to trade prior to receiving full information. Having many traders significantly complicates the theoretical analysis, so rather than providing a full equilibrium characterization as I do here, Kendall (2018) provides conditions under which market rushes occur that are sufficient for making predictions in the laboratory environment.[9]

[8]One can introduce a continuum of informed trader types based on a private discount factor or private cost of delay as in Harsanyi (1973) to convert the mixed strategies to threshold strategies based on type. The main results are unchanged in this variation.

[9]Kendall (2018) also differs in that the market maker is assumed to post only a single price rather than

The key ingredient of the model, that information improves with time, is shared by the contemporaneous papers of Bouvard and Lee (2015) and Dugast and Foucault (2017). In Dugast and Foucault (2017), traders choose between raw information, which may be completely uninformative, and perfect information. Competition creates a cost of acquiring perfect information when raw information is sufficiently precise, but when raw information is noisy, traders that process information benefit from waiting, not only because they receive better information, but also because they can exploit mispricing in the first period after learning that the initial traders were uninformed.[10] In Bouvard and Lee (2015), research produces better information about the private value of an asset (interpreted as firms performing risk management). Neither of these papers contrasts the time cost of research with a monetary cost, nor studies how the time cost varies with model primitives, as I do here.

Competition in the model takes place in the market itself, through the price mechanism. Alternatively, competition could take place in the information market. In Veldkamp (2006a,b), competitive markets for information generate prices which depend upon the number of investors buying the information, which in turn depend upon uncertainty in the asset value. Her model therefore also generates a dependence of the cost of information on uncertainty, but through a very different mechanism. Competition in the information market can also affect the value of information: Manela(2014) shows that the value of private information can actually increase when information diffuses faster (thereby becoming publicly available more quickly) because risk-averse traders are then able to realize their profits in advance of additional shocks.[11] In Manela (2014), private information is always acquired ahead of it becoming public. Here, instead, it is the fact that public information arrives while one is acquiring better private information that makes the arrival of public information costly.

In addition to the literature on information acquisition, this paper is related to the literature on rational panics which shares the intuition that waiting to trade is costly (see Romer (1993), Bulow and Klemperer (1994), Smith (1997), Lee (1998), Barlevy and Veronesi (2003), Brunnermeier and Pedersen (2005), and Pedersen (2009)). This literature emphasizes the role that rushing to trade plays in producing price crashes, whereas I show that it also affects the informational content of trades.

separate bid and ask prices. Also, traders always receive an initial signal and can obtain an additional signal by waiting, but are restricted to trade only once. These differences make the model more amenable to having many traders, but with a greater level of abstraction.

[10] Here, information improves with time, but is always informative, making the model more appropriate for the case of verified sources of information. The structure in Dugast and Foucault (2017) is more appropriate for the case of rumors that originate from unverified sources, such as social media.

[11] Another distinct means through which competition can affect incentives to acquire information is illustrated by Glode, Green, and Lowery (2012). In their bilateral bargaining model, competition increases the incentive to acquire information in order to avoid being in a disadvantaged bargaining position.

Preemption motives are also shared by the literature on real options with imperfect competition (e.g. Kulatilaka and Perotti (1998), Grenadier (2002), and Weeds (2002)). In real options models, as here, waiting is valuable and becomes more so as uncertainty increases, because it provides better information about investment opportunities. Here, however, when uncertainty is higher, competition unambiguously decreases the value to waiting through the greater impact of information revealed by competitive trades in the market. In real options models, instead, the effect of uncertainty depends upon the way in which competition is modeled in the post-investment market. Uncertainty may lead to more or less delay, whether in a winner-takes-all environment (Weeds(2002)) or when competition is via Cournot competition (Kulatilaka and Perotti (1998) and Grenadier (2002)). In a paper related to real options, Banerjee and Breon-Drish (2017) consider the decision of a strategic trader who can choose to condition her information acquisition decision on a publicly observable signal. In their novel setup, the asset value stochastically varies directly with a publicly observable signal, so that uncertainty in the asset's value can either increase or decrease with public information. When the information acquisition decision is observable, they show that the potentially informed trader waits until there is sufficient uncertainty in the asset value such that it is worth paying a monetary cost for private information. Here, instead, public information always reduces uncertainty in the asset value, imposing a cost of waiting.[12]

Finally, Malinova and Park (2014), focus on the strategic interaction between an informed trader and the market maker in a two-period setup similar to mine, showing that it can produce intraday trading patterns consistent with empirical data. The key difference in their model is that the informed investor is endowed with a fixed precision signal.

The paper is organized as follows. I begin with the model of a single trader in Section 2. With this model, I establish a benchmark for the value of information by considering a monetary cost of information (Section 2.3). I then study the time cost of information by replacing the monetary cost with a public information event (Section 2.4). I provide the main results of the paper in Section 3 by showing that competition introduces a time cost analogous to a public information event. In Section 4, I use the model of competition to show that the time cost of information causes the greatest reduction in information acquisition when it is most beneficial for informational efficiency. Finally, I derive testable empirical predictions in Section 5) and suggest ideas for future work in Section 6.

[12]Banerjee and Breon-Drish (2017) also consider the case in which the information acquisition decision is unobserved, showing that unobservability interestingly leads to inexistence of equilibrium, whether in continuous time or in discrete time as the length of a period goes to zero.

2 Single Trader Model

2.1 Description

There are two trading periods, $t \in \{0, 1\}$. A single asset of unknown value, $V \in \{0, 1\}$, with common prior, $p_0 = Pr(V = 1) \in (0, 1)$ is traded in each period. Its value is realized prior to the trading periods, but becomes public knowledge only at $t = 2$.[13] Trade takes place between a risk-neutral trader and risk-neutral market maker. As in Glosten and Milgrom (1985), all trades are for a fixed size, normalized to a single unit: either a purchase or a short sale.[14]

With probability $1 - \mu$, $\mu \in (0, 1)$, the trader is an uninformed trader who buys or sells with equal probability in either trading period. Uninformed traders represent traders that trade for exogenous reasons, such as their own liquidity needs, portfolio re-balancing, etc. Their presence prevents the adverse selection problem faced by the market maker from precluding all trade.

With probability μ, the trader is informed and can generate a private, binary signal, the quality of which is increasing in the amount of time spent doing research (and is otherwise costless). Due to the discrete nature of the model, the signal qualities are also discrete. In particular, if the trader invests little time doing research, she receives a private, binary signal at $t = 0$, $s_0 \in \{0, 1\}$, which correctly identifies V with probability $q_0 = Pr(s_0 = 1|V = 1) = Pr(s_0 = 0|V = 0) \in (\frac{1}{2}, 1)$. In this case, I say she *rushes*. If, instead, the trader invests an additional time period doing research, she receives a private, binary signal at $t = 1$, $s_1 \in \{0, 1\}$, which identifies V with the larger probability $q_1 = Pr(s_1 = 1|V = 1) = Pr(s_1 = 0|V = 0) \in (q_0, 1]$.[15] In this case, I say she *waits*. There are no monetary costs associated with obtaining either signal, and the information acquisition decision is unobserved by the market maker.

I denote the trader's action in each period, $a_t \in \{buy, sell, no\ trade\}$, and I make two restrictions to simplify the exposition and analysis. The first is that a trader may not place a trade in the first period if she waits. This assumption is innocuous because trading prior to receiving private information earns at most zero in expectation.[16] The second restriction

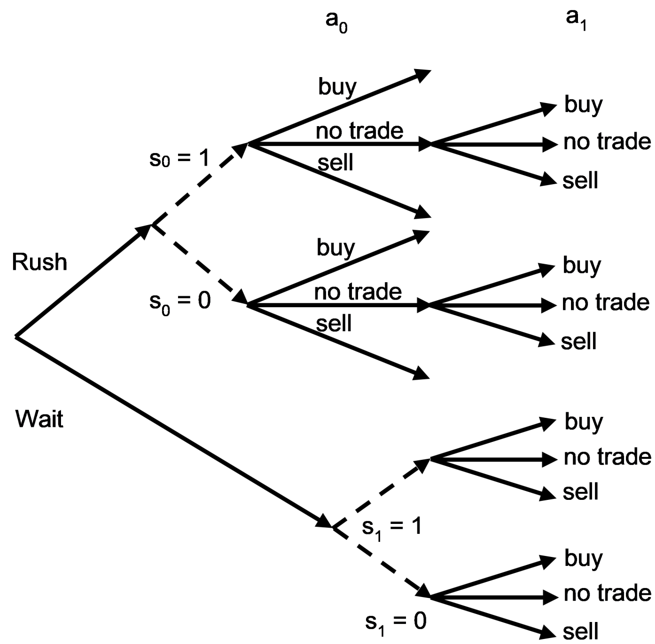
[13]This assumption is standard in models of informed trading. See Glosten and Milgrom (1985), Kyle (1985), Admati and Pfleiderer (1995), and Foster and Viswanathan (1995), among others.

[14]This assumption is also standard in models of informed trading with risk-neutral traders. In addition to Glosten and Milgrom (1985), see Cipriani and Guarino (2014), Dugast and Foucault (2017), and Malinova and Park (2010,2014).

[15]I impose the restriction, $q_1 > q_0$, because it is natural to assume that information improves with time. The nature of the time cost is the same when $q_1 \leq q_0$, but in this case, when the probability of informed trading becomes small, no trade-off exists and the informed trader rushes with probability one.

[16]Recall that the informed trader receives either the first period or second period signal (not both), so if

Figure 1: Decision Tree for the Informed Trader



is that if she acquires information in the first period, she may only trade once (in either period). Formally, $a_1|a_0 \in \{buy, sell\} = no\ trade$. I argue in Part B of the Online Appendix that the main insights of the model are unchanged if this restriction is relaxed. Figure 1 provides a simple representation of the decision tree faced by the informed trader.

Between the two trading periods, information about the asset's value becomes public. This information may most simply be public news, but it could also be information revealed by others' trades, a possibility I consider explicitly in Section 3. To allow for both possibilities, let e denote a generic public event. I assume that the event: (i) has at least two possible realizations from the set, $E = \{e^1, e^2, \ldots, e^n\}$, (ii) is informative, $Pr(e = e^i|V = 1) \neq Pr(e = e^i|V = 0)$, for at least one realization, and (iii) is symmetric, $Pr(e = e^i|V = 1) = Pr(e = e^j|V = 0)$ for some j, for all realizations, e^i. Information contained in the public event and the private signals are assumed to be independent conditional on the asset value.

As is standard (Glosten and Milgrom (1985)), the market maker is assumed to face (unmodeled) perfect competition, earning zero expected profit. He accounts for the private information contained in the current order and posts separate bid and ask prices, B_t and A_t, at which he is willing to buy and sell, respectively. Let I_t denote the information set

she waits, she has no private information in the first period. She would therefore only be willing to trade in the first period if the bid-ask spread were zero, and allowing her to do so does not otherwise change the equilibrium.

of publicly available information at time t, including trades in previous periods and, at $t = 1$, information revealed by the public event. If the informed trader rushes and then trades at time $t \in \{0, 1\}$, her expected profit is given by $E[V|I_t, s_0] - A_t$ if she buys, and $B_t - E[V|I_t, s_0]$ if she sells. If she waits, the corresponding expressions are $E[V|I_1, s_1] - A_1$ and $B_1 - E[V|I_1, s_1]$.

2.2 Preliminaries

The solution concept is Perfect Bayesian Equilibrium. It is convenient to denote the public expectation of the asset's value, $p_t = E[V|I_t] = Pr[V = 1|I_t]$. Due to the assumption that the market maker earns zero expected profit, it is easily shown that the bid and ask prices in each period are given by the expected value of the asset conditional on public information and the information contained in the current order, $B_t = E[V|I_t, a_t = \mathit{sell}]$ and $A_t = E[V|I_t, a_t = \mathit{buy}]$. These posted prices depend upon the market maker's belief that a trade contains information, which, due to the possibility of uninformed traders, is pinned down by Bayes' rule at all possible histories. As usual, equilibrium requires that beliefs are correct.

Given the posted prices, the informed trader maximizes her expected profit. Lemma 1 first characterizes the optimal trading strategy for any possible information acquisition strategy. All proofs are contained in the Appendix.

Lemma 1: *In any equilibrium, an informed* trader *buys with a positive signal ($s_t = 1$) and sells with a negative signal ($s_t = 0$).*

The result of Lemma 1 is standard and intuitive: an informed trader with a positive signal buys because her private belief exceeds the ask price, and conversely. The presence of uninformed traders ensures a gap between the posted price and the informed trader's private belief, allowing her to make a positive expected profit from trading. The market maker breaks even, profiting from the uninformed traders and losing to the informed.

Accounting for her optimal trading strategy, the informed trader chooses the quality of research to undertake to maximize her expected profit. I denote the (possibly degenerate) probability that the she rushes to acquire information at $t = 0$, $\hat{\beta}$. The probability that she waits is then $1 - \hat{\beta}$. I begin by assuming that the informed trader trades in the first period after rushing. Although she could instead delay her trade to the second period (or mix between the two periods), in each of the results that follow, I show that (provided that the probability of informed trading, μ, is not too large) she always prefers to trade immediately (see Part B of the Online Appendix for a discussion of the case in which μ is large).

Under this assumption, an equilibrium is fully characterized by $\hat{\beta}$, along with the trading strategy of Lemma 1. In this case, her expected profit, π_0, from rushing is given by

$$\begin{aligned} \pi_0(\beta, p_0, q_0) &= Pr(s_0 = 1)\left(Pr(V = 1|s_0 = 1) - A_0\right) \\ &+ Pr(s_0 = 0)\left(B_0 - Pr(V = 1|s_0 = 0)\right) \end{aligned} \tag{1}$$

where the bid and ask prices are

$$A_0 = \frac{p_0 Pr(a_0 = buy|V = 1)}{p_0 Pr(a_0 = buy|V = 1) + (1 - p_0)Pr(a_0 = buy|V = 0)}$$

and

$$B_0 = \frac{p_0 Pr(a_0 = sell|V = 1)}{p_0 Pr(a_0 = sell|V = 1) + (1 - p_0)Pr(a_0 = sell|V = 0)}$$

respectively. The bid and ask prices depend upon the market maker's belief, β, about the informed trader's information acquisition strategy, $\hat{\beta}$, through the probabilities of observing a buy or sell: $Pr(a_0 = buy|V = 1) = Pr(a_0 = sell|V = 0) = \mu q_0 \beta + m$ and $Pr(a_0 = buy|V = 0) = Pr(a_0 = sell|V = 1) = \mu(1 - q_0)\beta + m$, where $m = \frac{1-\mu}{4}$ is the probability of observing a buy (or sell) order from an uninformed trader in a given period. The first term in (1) corresponds to the profit from buying if the informed trader receives a positive signal, and the second from selling with a negative signal. In the Appendix, I show that we can use Bayes' rule and the symmetry of the buy and sell decisions to obtain

$$\pi_0(\beta, p_0, q_0) = \omega_0 m(2q_0 - 1)\left(\frac{1}{Pr(a_0 = buy)} + \frac{1}{Pr(a_0 = sell)}\right) \tag{2}$$

where $\omega_0 = Var(V) = p_0(1 - p_0)$ is the prior variance in the asset's value and $Pr(a_0 = buy) = p_0\left(\mu q_0 \beta + m\right) + (1 - p_0)\left(\mu(1 - q_0)\beta + m\right)$ is the probability of observing a buy decision in the first period ($Pr(a_0 = sell)$ is similar).

If no public information arrives, the expected profit from waiting, denoted $\pi_1^{NP}(\beta, p_0, q_1)$, is very similar to that from rushing:[17]

$$\pi_1^{NP}(\beta, p_0, q_1) = \omega_0 m(2q_1 - 1)\left(\frac{1}{Pr(a_1 = buy)} + \frac{1}{Pr(a_1 = sell)}\right) \tag{3}$$

[17]In deriving (3), note that observing no trade in the first period generally provides information about whether or not the trader is informed, but not about V itself. If a trader rushes and delays her trade with a probability that depends upon her private information, then no trade may (partially or fully) reveal her private information, but this information becomes redundant upon observing the trade order (Lemma 1). It is for this reason that the bid and ask prices in the second period (and thus (3)) do not contain expressions involving the probability of no trade.

$Pr(a_1 = buy) = p_0\,(\mu q_1(1-\beta) + m) + (1-p_0)\,(\mu(1-q_1)(1-\beta) + m)$ is the probability of observing a buy decision in the second period ($Pr(a_1 = sell)$ is similar). If public information arrives, the informed trader accounts for its expected impact on prices in the second period. If we define the expected value of the asset conditional on realization e^i of the public event as p_{e^i}, then the expected profit from waiting conditional on e^i is given by $\pi_1^{NP}(\beta, p_{e_i}, q_1)$, where p_{e^i} replaces p_0 in (3). Taking the expectation over the possible realizations of the public event results in the overall expected profit:

$$
\begin{aligned}
\pi_1(\beta, p_0, q_1) &= \sum_{e^i \in E} Pr(e_i)\pi_1^{NP}(\beta, p_{e_i}, q_1) \\
&= \omega_0 m(2q_1 - 1) \sum_{e^i \in E} \left(\frac{Pr(e = e^i|V=0)Pr(e = e^i|V=1)}{Pr(a_1 = buy, e = e^i)} \right. \\
&+ \left. \frac{Pr(e = e^i|V=0)Pr(e = e^i|V=1)}{Pr(a_1 = sell, e = e^i)} \right) \qquad (4)
\end{aligned}
$$

We can now determine the value of additional information, by the difference in expected profits it generates.

2.3 The Value of Information

I establish the value of information by deriving a benchmark result for the case in which information is acquired at a monetary cost. In particular, I consider how the value varies as a function of the prior, p_0, which maps one-to-one to the prior uncertainty in the asset value, $\omega_0 = p_0(1-p_0)$. Intuition suggests that when uncertainty is high, private information of fixed quality is more valuable because it results in a larger difference between a trader's belief and the public belief, allowing her to earn a larger expected profit. I confirm this intuition in Lemma 2: given either period's information quality, the informed trader's expected profit peaks at the highest value of uncertainty (which occurs when $p_0 = \frac{1}{2}$). I also establish a symmetry property which arises from the fact that the profit from obtaining a particular signal realization at a prior of p_0 is the same as that from obtaining the opposite realization at a prior of $1 - p_0$.

Lemma 2: *The expected profits at $t = 0$ and $t = 1$ are both symmetric and concave in the prior: they peak at maximum prior uncertainty, $p_0 = \frac{1}{2}$, and decrease to zero at $p_0 = \{0, 1\}$.*

Lemma 2 suggests that the informed trader would be willing to pay the most for information when $p_0 = \frac{1}{2}$. In fact, this result has been previously established in models with monetary costs. In a sequential trading model with a fixed cost of information, Nikandrova (2014) shows that traders are only willing to pay for information when p_0 is near $\frac{1}{2}$. Lew (2013) instead allows traders to pay to increase the probability that they learn the asset value for certain. With a cost that is quadratic in this probability, he also shows that traders acquire the most information when $p_0 = \frac{1}{2}$.

Here, traders face a choice between signals of different quality, unlike in the papers by Nikandrova (2014) and Lew (2013). The question is then whether or not the *marginal* value of the better signal peaks at $p_0 = \frac{1}{2}$. Perhaps surprisingly, the answer is - not always. The reason is that this value also depends upon the difference between trading costs due to differences in the (endogenous) bid-ask spreads across periods. In Propositions 1 and 2, I impose Assumption 1, which ensures that the effect of the bid-ask spread is limited.

Assumption 1: *The probability of informed trading is not too large, $\mu \leq \bar{\mu}$.*

In Lemma A4 of the Appendix, I show that the strictly positive upper limit required for Assumption 1, $\bar{\mu} > 0$, exists. In Part B of the Online Appendix, I show numerically that $\bar{\mu}$ can be very large, even one for most parameterizations. There, I also discuss the model's predictions when μ is too large for the results of Propositions 1 and 2 to hold.

To understand the marginal value of information, consider a setup identical to that described in Section 2.1 except that, instead of public information arriving while a trader waits, the trader faces a cost of delay, $c \in (0, 1)$, which must be paid if she trades in the second period. This cost may reflect an opportunity cost, for example.[18] If the informed trader is willing to pay this fixed cost, then the value of trading in the second period must be higher.

The profits $\pi_0(\beta, p_0, q_0)$ and $\pi_1^{NP}(\beta, p_0, q_1)$ depend upon the market maker's belief, β, about the information acquisition strategy of the informed trader. In equilibrium, his belief must coincide with the trader's actual information acquisition strategy, $\beta = \hat{\beta}$. Denoting the equilibrium probability of rushing for the fixed cost case, β^{C*}, an equilibrium must satisfy

$$\begin{array}{rcllrcl}
\beta^{C*} & = & 1 & if & \pi_1^{NP}(1, p_0, q_1) - c - \pi_0(1, p_0, q_0) & \leq & 0 \\
\beta^{C*} & \in & (0,1) & if & \pi_1^{NP}(\beta^{C*}, p_0, q_1) - c - \pi_0(\beta^{C*}, p_0, q_0) & = & 0 \\
\beta^{C*} & = & 0 & if & \pi_1^{NP}(0, p_0, q_1) - c - \pi_0(0, p_0, q_0) & \geq & 0
\end{array}$$

[18] I could instead assume the cost is only paid when a trader obtains a high quality signal, but to be consistent with the public information model, I assume the cost is paid even if one acquires the low quality signal and then delays trading to the second period. In this way, the models are identical except that the cost of obtaining a better quality signal is a fixed, monetary cost, rather than the cost of forgone profits due to a public event.

For many parameterizations, β^{C*} is interior: intuitively, the higher the probability that the market maker places on informed trade in a period, the larger the spread between the bid and ask prices must be. An increase in the spread in turn reduces the profit from trading, causing the trader to shift trade to the other period. This argument suggests a unique equilibrium, which I confirm in Proposition 1 below.

Proposition 1 also establishes how the marginal value of information varies as a function of the prior uncertainty: it peaks at highest uncertainty so that traders are willing to pay the most for better information (wait most) at this time.[19]

Proposition 1: *With a monetary cost of information:*
a) A unique equilibrium exists in which the informed trader rushes with probability, $\beta^{C*} \in [0,1]$*, trades in the period she acquires information, and buys or sells as per Lemma 1.*
b) *The informed trader acquires the better signal most often when the prior uncertainty is largest:* β^{C*}*, is a minimum when uncertainty is highest,* $p_0 = \frac{1}{2}$*, and weakly increases as* $p_0 \longrightarrow \{0,1\}$*. The increase is strict whenever* $\beta^{C*} \in (0,1)$*.*

2.4 The Time Cost of Information

I now return to the model described in Section 2.1 to study how public information arrival between trading periods creates a time cost of performing research. I first establish that public information does in fact impose a cost by showing in Lemma 3 that it strictly reduces the expected profit from acquiring better information.

Lemma 3: *The expected profit from waiting decreases with the arrival of public information,* $\pi_1(\beta, p_0, q_1) < \pi_1^{NP}(\beta, p_0, q_1)$*.*

Lemma 3 stems from the fact that the value of additional information is proportional to prior uncertainty in the asset value, and public information reduces this uncertainty. This same effect is present in the model of Holden and Subrahmanyam (1992) who show that information released by others' trades causes traders to act more quickly on their own information. However, in their model, traders know the asset value perfectly so that acting more quickly does not affect the quality of private information produced. When traders face a trade-off in terms of the quality of information, we can ask when the effect of the information release on incentives to do time-consuming research is greatest. Proposition 2 provides the answer.

[19]Specifically, the informed trader pays a monetary cost of $c(1-\beta^{C*})$ which decreases in β^{C*}, peaking at maximum uncertainty.

Proposition 2: *When public information generates a time cost of information:*
a) A unique equilibrium exists in which the informed trader rushes with probability, $\beta^* \in [0, 1]$*, trades in the period she acquires information, and buys or sells as per Lemma 1.*
b) *The equilibrium probability that the informed trader rushes,* β^**, is largest when uncertainty is highest,* $p_0 = \frac{1}{2}$*, and weakly decreases as* $p_0 \longrightarrow \{0, 1\}$*. The decrease is strict whenever* $\beta^* \in (0, 1)$*.*

Comparing Propositions 1 and 2, we see a stark difference between a model with monetary costs and one with time costs due to public information arrival. Although a trader is most willing to pay a monetary cost when uncertainty is largest ($p_0 = \frac{1}{2}$), due to the nature of the time cost of public information, she acquires better information *least* often under this same condition. This result derives from the fact that the reduction in uncertainty due to the public information release is more consequential when prior uncertainty is higher. In the case of very low uncertainty, $p_0 \to \{0, 1\}$, public information reveals almost nothing new and therefore changes prices and the expected profit from waiting only negligibly. On the other hand, when p_0 is close to $\frac{1}{2}$, the decrease in expected profit due to a public event is large. Alternatively, we can think of the impact in terms of price movements. Due to unconditional correlation between the trader's information and that which becomes public, in expectation prices move in the direction that reduces the trader's profit. When p_0 is near one half, the public event causes market prices to move substantially, whereas prices move very little when uncertainty is low.

These contrasting results show that, although it is possible to interpret the monetary cost in a standard model as a time cost, when we model the time cost explicitly, we see that it must take a very non-standard form. Rather than simply being a weakly increasing, convex function of signal quality (see, for example, Grossman and Stiglitz (1980) and Verrecchia (1982)), the cost function must instead depend upon prior uncertainty in the asset value. To see this explicitly, we can calculate the monetary cost equivalent of the public information event: the cost that induces the same probability of rushing. Provided the equilibrium probability of rushing, β^*, is interior, this cost is defined by $c^P(p_0, q_0, q_1) \equiv \pi_1^{NP}(\beta^*, p_0, q_1) - \pi_0(\beta^*, p_0, q_1)$.[20] Proposition 3 establishes that c^P monotonically increases with uncertainty.

[20]When β^* is a corner solution, the monetary cost is bounded, but not uniquely defined. For ease of exposition, I focus on the interior case.

Proposition 3: *If the equilibrium probability of rushing with public information arrival is interior, $\beta^* \in (0,1)$, then the monetary cost, $c^P(p_0, q_0, q_1)$, required to induce the same probability of rushing is largest when uncertainty is highest, $p_0 = \frac{1}{2}$, and strictly decreases to zero as $p_0 \longrightarrow \{0,1\}$.*

3 Competitive Model

The single trader model of the previous section is the simplest possible setting in which we can study the time cost of information and contrast it with the standard monetary cost assumption. However, because in this model public information crowds out private information acquisition, the overall informational efficiency of the market depends upon which effect dominates. In this section, I show that competition among traders similarly leads to time costs - intuitively, trades that reveal information are simply another form of public information. Importantly, when competition creates incentives to rush, it unambiguously reduces the amount of private information produced, which I show in Section 4 generally reduces informational efficiency.

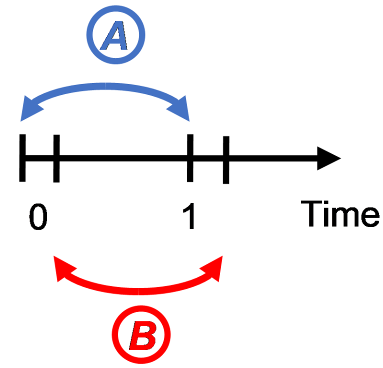
To study competition, consider a setting in which two traders (A and B) can acquire private information over time. Trader B arrives to the market during the time trader A does research, resulting in the overlapping structure shown in Figure 2.[21] Rather than describe the model in terms of four periods, to preserve notation, I continue to label the periods 0 and 1, but divide each period into two subperiods, with trader A having the first opportunity to trade in each. In each period, after a trade by the first trader, the market maker updates his beliefs and posts new bid and ask prices for trader B.

I allow trader B to arrive with probability $\alpha \in (0,1]$, which serves as a measure of the level of competition.[22] The structure of the asset's value and the signals available to each trader are identical to the single trader model. Signals are conditionally independent across traders. The probabilities that trader A and trader B are informed are μ_A and μ_B, respectively. As

[21] An overlapping structure, as opposed to allowing multiple traders to trade simultaneously, simplifies the model by removing the need for the market maker to update the bid and ask prices when simultaneous orders are received (see Malinova and Park(2014)). However, the overlapping structure is not necessary for competition to induce rushing. Kendall (2018) provides a model in which all traders arrive at time zero and, under some conditions, simultaneously rush to trade. There, I avoid the multiple order problem by assuming the market maker only infers information from past, but not current, orders.

[22] Note that the bid and ask prices do not depend on α because, upon observing a trade in the second subperiod of a period, the market maker directly learns of the second trader's presence. That is, I assume that the market maker has full knowledge of the timing of the game and so can always identify the owner of a trade.

Figure 2: Overlapping Timing Structure



Note: Trader identifies are shown in circles. Time periods are identified below the time line.

in the single trader model, I define the probabilities of an uninformed buy or sell by each trader as $m_A \equiv \frac{1-\mu_A}{4}$ and $m_B \equiv \frac{1-\mu_B}{4}$. Trader identifiers on actions, strategies, and profits are denoted by subscripts and, when necessary, precede the time subscript (e.g. the action of trader i in period t is denoted $a_{i,t}$).

As in the single trader model, I impose an assumption on the level of the probability of informed trading for the results in Proposition 4. Assumption 2 is the equivalent of Assumption 1.

Assumption 2: *The probability of informed trading is not too large, $\mu_A, \mu_B \leq \bar{\mu}$.*

My primary focus is on the behavior of trader A who must be concerned that trader B may arrive and preempt her trade. However, note that trader A also impacts trader B when both wait. I assume trader B observes whether or not trader A trades (but not whether or not she is informed) and therefore knows when she can freely do research without any possible intervening trade.[23] I study the Perfect Bayesian Equilibrium in which each trader accounts for the actions of the other in choosing her information acquisition and trading strategies.

For each trader, the potential informed trade by the other trader is a form of a public event so that many of the results of the previous sections apply directly to the model with competition. In particular, Lemma 1 applies to the trading strategies of both traders and, assuming each trader trades immediately after acquiring information (which I later show), each has an expected profit from rushing given by equation (2).

On the other hand, each trader's expected profit from waiting requires modification. If trader A waits, then she must account for the probability that trader B exists, α, and, if so, the probability, β_B, that she believes trader B rushes. If trader B doesn't exist,

[23] If she can't observe the trade directly, she can infer it from the resulting change in prices.

then trader A's profit is the same as if she were alone in the market, $\pi_1^{NP}(\beta_A, p_0, q_1)$. We can thus write trader A's expected profit from trading in period 1 as $\Pi_{A,1}(\beta_A, \beta_B, p_0, q_1) = \alpha\pi_{A,1}(\beta_A, \beta_B, p_0, q_1) + (1-\alpha)\pi_1^{NP}(\beta_A, p_0, q_1)$. $\pi_{A,1}(\beta_A, \beta_B, p_0, q_1)$ is given by (4) and depends on β_B through the probabilities of three possible public events corresponding to the possible trades by trader B: buy, sell, or no trade.[24]

If trader B observes trader A trade, she rushes with probability β^{NP*}, defined to be the probability a trader rushes when no public information event occurs. On the other hand, if trader B observes no trade, then she updates her belief about the probability trader A is informed, which makes her expected profit from waiting:

$$\begin{aligned} \pi_{B,1}(\beta_A, \beta_B, p_0, q_1) &= \frac{\omega_0 m_B(2q_1 - 1)}{\mu_A(1-\beta_A) + 2m_A} \sum_{e^i \in E} \left(\frac{Pr(e^i|V=0)Pr(e^i|V=1)}{Pr(a_{B,1} = buy, e = e^i)} \right. \\ &+ \left. \frac{Pr(e^i|V=0)Pr(e^i|V=1)}{Pr(a_{B,1} = sell, e = e^i)} \right) \end{aligned} \tag{5}$$

β_A enters $\pi_{B,1}(\beta_A, \beta_B, p_0, q_1)$ directly in the denominator and also through the probabilities of two public events, a purchase or sale by trader A.

Given expressions for the expected profits of each trader, Proposition 4 first establishes that a unique equilibrium exists in which each trader trades in the period she acquires information, and then characterizes several properties of this equilibrium.

[24] This set of public events is consistent with the assumptions made about public events in Section 2.1. The buy and sell events are informative and symmetric ($Pr(a_{B,0} = buy|V=1) = Pr(a_{B,0} = sell|V=0)$ and vice versa). The no trade event is uninformative and symmetric ($Pr(a_{B,0} = no\,trade|V=1) = Pr(a_{B,0} = no\,trade|V=0)$)=(1-$\alpha$) + $\alpha(\mu\beta_B + m_B)$.

Proposition 4: *In the competitive model:*

a) A unique equilibrium exists in which each trader trades in the period she acquires information, and buys or sells as per Lemma 1. Trader A rushes with probability, $\beta_A^* \in [0,1]$. Trader B *rushes with probability,* $\beta^{NP*} \in [0, \frac{1}{2})$, *when trader A rushes, and* $\beta_B^* \in [0,1]$ *when trader A waits.*

In the unique equilibrium:

b) Both traders rush more often due to competition, $\forall \alpha > 0$, $\beta_A^*, \beta_B^* \geq \beta^{NP*}$. *Trader A rushes strictly more often,* $\beta_A^* > \beta^{NP*}$, *whenever* $\beta^{NP*} > 0$.

c) As the probability that trader B arrives increases, trader A rushes weakly more often, $\frac{d\beta_A^*}{d\alpha} \geq 0$, *with strict inequality whenever* $\beta^{NP*} > 0$ *and* $\beta_A^* \in (0,1)$.

d) The probability that trader A rushes, β_A^*, *is largest when uncertainty is highest,* $p_0 = \frac{1}{2}$, *and weakly decreases as* $p_0 \longrightarrow \{0,1\}$. *The decrease is strict whenever* $\beta_A^* \in (0,1)$.

e) The probability that trader A rushes is strictly increasing in the first period signal strength, q_0, *whenever the probability is interior:* $\frac{d\beta_A^*}{dq_0} > 0 \ \forall \beta_A^* \in (0,1)$.

Uniqueness of the equilibrium in Proposition 4 is not immediate because there is strategic complementarity in the timing decisions from the point of view of trader A. As trader B waits more often, it reduces the impact on trader A, causing her to best respond by waiting more often. Uniqueness arises because of strategic substitutability from the point of view of trader B: as trader A waits more often, it increases the impact on trader B causing her to best respond by rushing more often.[25]

Part b of Proposition 4 shows that the competition among traders causes both to rush more often than if alone in the market, so that, in the aggregate, less information is acquired.[26] Part c) shows that trader A rushes more often as trader B's probability of arrival increases. Intuitively, the increased chance of a price impact makes it less profitable to wait. Given that trader B impacts trader A, we expect from Proposition 2 that trader A rushes most often when uncertainty is highest. This result does not follow immediately as a corollary to Proposition 2 though, because we must account for the fact that trader B's strategy also changes with uncertainty. Nevertheless, part d) of Proposition 4 establishes that the impact of competition is indeed greatest when uncertainty is highest. Finally, part

[25]The nature of strategic substitutability and complementarity are very different here from in static models such as Grossman and Stiglitz (1980). In a static model, the strategic interaction in information acquisition comes about through simultaneous learning through prices due to rational expectations. Here, instead, the strategic interaction is through the dynamic price impacts (or lack thereof) of others' trades.

[26]Numerical simulations suggest that trader B also rushes *strictly* more often when trader A is present than when not, but, analytically, I have not been able to rule out the possibility that trader A may rush with probability one, allowing trader B to mix as in the case in which trader A is not present.

e) provides an intuitive comparative static for the initial signal quality which provides the basis for an empirical prediction in Section 5. As q_0 increases, trader A rushes more often both because she receives a larger profit if she rushes and because trader B's price impact becomes larger if she waits.[27]

The main results of this section and the previous one are driven by the assumptions that better information arrives over time and traders must choose which quality of information to acquire. In Part B of the Online Appendix, I provide further results and extensions. First, I elaborate on the assumption that a trader can only acquire one signal or the other. I discuss situations in which the assumption directly applies, as in a one-time, irreversible investment (I show that the results hold when a trader can only trade in one direction), and then discuss relaxing the assumption. Second, I show numerically that the limit on the upper probability of informed trading required for the main results is generally large, larger than the most conservative empirical estimates, and also discuss the model's predictions when this upper limit is exceeded. Third, I show that the necessary features of the model are also present in a model with normally distributed asset values and signals so that the results are not driven by their binary nature. Finally, I argue that the results are robust to allowing a trader that rushes to trade in both the first and second periods, and that the results are not driven by the presence of a bid-ask spread or the market power of the traders.

4 Informational Efficiency

Traders' incentives to acquire private information matter because they affect the informational efficiency of market prices. To study informational efficiency, I measure it in the standard way, by the pricing error, $\mathbb{E}_2 \equiv E\left[(V - p_2)^2\right]$, where p_2 is the public belief after both trading periods (but prior to the asset value becoming known).[28] A larger pricing error corresponds to less informative prices.

The proof of Proposition 5 develops an expression for the pricing error and shows that, for a given information acquisition strategy of a single trader, the reduction in pricing error ($\mathbb{E}_0 - \mathbb{E}_2 = \omega_0 - \mathbb{E}_2$) is greatest when $p_0 = \frac{1}{2}$.

[27]It is possible to establish this comparative static in the case of a single trader as well, but I omit it for brevity. The comparative static with respect to the second period signal quality is more cumbersome in the competitive case, but it is easy to show that the trader waits more often as q_1 increases in the single trader case. Numerical simulations suggest that the comparative statics with respect to the probabilities of informed trading, μ_A and μ_B, are non-monotonic.

[28]The probability a trader is informed, μ, is exogenous in the model, so that I do not study the effect on informational efficiency of traders choosing to become informed or not. Instead, I study the effect of traders deciding *how much* information to acquire.

Proposition 5: *Fix the information acquisition strategy of a single trader, $\beta \in [0, 1]$. The reduction in the pricing error, $\omega_0 - \mathbb{E}_2$, due to the information revealed by her trades is strictly largest when uncertainty is highest ($p_0 = \frac{1}{2}$).*

Given the fact that the informed trader rushes most often when uncertainty is highest (Proposition 2), Proposition 5 shows that she forgoes information precisely when it is most valuable for improving the long-term informational efficiency of prices. The combination of these results suggests that rushing will have the greatest impact on informational efficiency at high uncertainty. However, this intuition can be offset by a second force which acts through the bid-ask spread: the more concentrated informed trade is within a single period, the greater the information revealed to the market and the better the informational efficiency.[29] Thus, it is conceivable that more rushing actually improves informational efficiency if this 'trade concentration' effect dominates the 'weaker information' effect.

To study which effect dominates as a function of the parameters of the model, I consider the model with competition of the previous section. The main advantage of using this model over the single trader model is that all information is endogenously generated.[30] In order to understand the impact of rushing on informational efficiency, one needs an appropriate benchmark, because the pricing error directly increases with prior uncertainty regardless of trading behavior. As a benchmark, I imagine a situation in which both traders ignore their competition and follow the information acquisition strategies they would use if alone in the market. The question is then, how does the pricing error change relative to the benchmark as we vary the parameters of the model? Comparative statics on the pricing error are difficult to perform analytically, so I rely on numerical simulation. Figure 3 plots the increase in the pricing error in equilibrium over that in the benchmark model. Each graph is a contour plot of the difference as a function of the prior, p_0, and the second period signal quality, q_1, holding fixed the values of initial signal quality, q_0, and probability of informed trading, $\mu_A = \mu_B$.

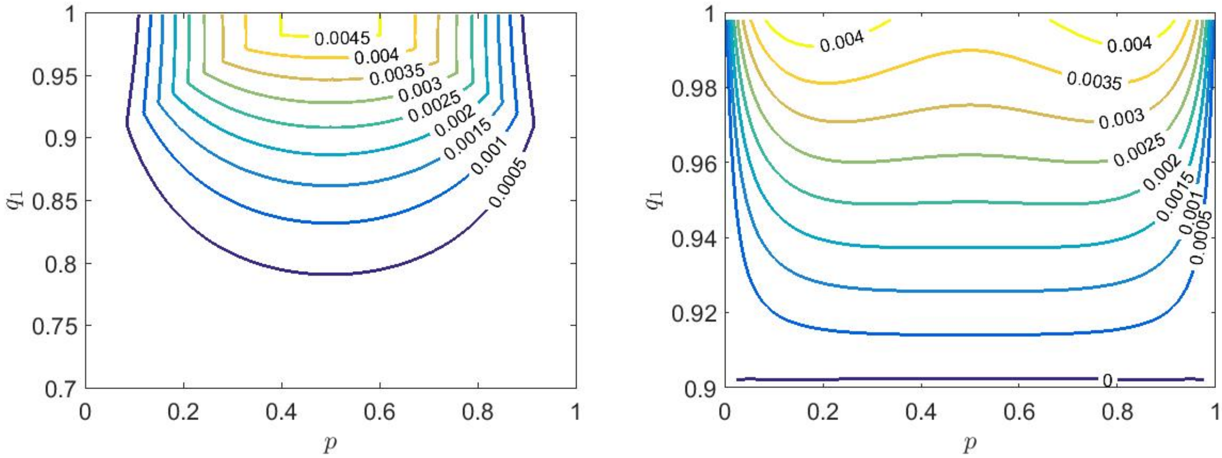
In the left graph of Figure 3, I consider a representative case in which q_0 and $\mu_A = \mu_B$ are not extreme ($q_0 = 0.7$ and $\mu_A = \mu_B = 0.42$).[31] In this case, which is typical of most parameterizations, the pricing error is always larger than in the benchmark model and increases with uncertainty: the 'weaker information' effect dominates the 'trade concentration'

[29] Intuitively, the informed trader is mixing precisely in order to 'hide' her information from the market maker, but this masking of information reduces informational efficiency.

[30] Studying informational efficiency in the single trader model requires one to explicitly model the public event which introduces at least one additional parameter. Furthermore, the choice of this parameter affects both the incentive to rush and informational efficiency directly.

[31] $\mu_A = \mu_B = 0.42$ corresponds to the estimate of Cipriani and Guarino (2014). See part B of the Online Appendix for more details.

Figure 3: Informational Efficiency



Note: Each graph provides a contour map of the increase in the pricing error when traders rush with their equilibrium probabilities over that when they rush as if alone in the market. For the left graph, $q_0 = 0.7$ and $\mu_A = \mu_B = 0.42$, and for the right graph, $q_0 = 0.9$ and $\mu_A = \mu_B = 0.85$.

effect.[32] Although typical, this result is not universal: the right graph provides an extreme example in which both q_0 and μ are large ($q_0 = 0.9$ and $\mu_A = \mu_B = 0.85$). In this case, when the second period signal quality, q_1, is just slightly larger than that of the first, q_0, rushing can actually improve informational efficiency (the difference in pricing error is slightly negative) by concentrating trade in the first period. In addition, when the second period signal strength approaches one, although rushing reduces informational efficiency, it does not do so most at highest uncertainty. The reason is, as I discuss in Part B of the Online Appendix, for extreme parameters, rushing probabilities do not necessarily peak at highest uncertainty. I summarize the numerical findings in Result 1.

> **Result 1:** *For typical parameters, informational efficiency is lower with competition than without, and the reduction is largest when uncertainty is highest.*

5 Empirical Implications

The model makes predictions about the private information traders acquire, but private information is typically unobservable so that the main results are not likely to be directly testable. To overcome this problem, I derive observable implications of the model and contrast them with a comparable model with monetary costs. The purpose of this section is not

[32]Losses are moderate in this example, with a maximum increase in the pricing error of just over 3%. However, in markets with many competitors, one could imagine larger informational losses as waiting for better information would be much more costly.

to develop tests that control for all possible confounds, but rather to guide such empirical tests, and to interpret existing empirical results in the context of the model.

To fixate ideas, consider the period of time after an announcement, such as earnings announcement.[33] If one assumes that public news releases may be differentially interpreted (as in Holthausen and Verrecchia (1990), Indjejikian (1990), and Kandel and Pearson (1995)), then processing news over time generates private information.[34] The first period in the model corresponds to the time immediately after the announcement.[35] For comparison purposes, I contrast the predictions of the competitive model of Section 3 with those of the model in Section 2.3 in which a trader, rather than facing competition, pays a monetary (opportunity) cost of trading in the second period.

The first observable for which the model makes unambiguous predictions is the order flow imbalance: the difference in buy vs. sell-initiated orders.[36] Intuitively, a larger imbalance suggests traders have better private information: with completely noisy information, buys and sells are equally likely, but with perfect information, all informed trades are in a single direction.

As in Section 3, I continue to focus on the first trader. For this trader, the expected difference between the number of buys and sells at the end of the second period is simply, $Pr(a_0 = buy) + Pr(a_1 = buy) - Pr(a_0 = sell) - Pr(a_1 = sell)$. Because positive and negative differences are equally informative, I consider the absolute value of this measure, $|E[IB]| \equiv |Pr(a_0 = buy) + Pr(a_1 = buy) - Pr(a_0 = sell) - Pr(a_1 = sell)|$.[37] This measure is zero when buys and sells are equally likely and increases to one if all trades are purchases or sales, and so is comparable to the net order flow in actual market data. Accounting for

[33]Alternatively, we could think about incentives to acquire information before an announcement. Weller (2018) shows that less private information is acquired prior to an earnings announcement when there is more algorithmic trading present. His finding is consistent with the model in that algorithmic trading more rapidly incorporates public information into prices, making the cost of delaying to acquire better quality private information larger.

[34]Li (2015) provides evidence that market prices do not immediately incorporate the public information in earnings announcements, and that it can be profitable to rush based on weak information. He shows that following a simple rule of buying a stock as soon as possible if both the revenue and earnings per share targets are met, and selling if neither is met, beats the market by 11.5% per year after costs.

[35]For the length of a period, I have in mind times on the order of seconds, minutes, or perhaps hours. For information that takes longer (days) to produce, factors in addition to price movements are likely to play an important role in information acquisition decisions, such as opportunity costs.

[36]For several common observables, competition and uncertainty have a countervailing effect to the direct effect of information quality, so that the model's predictions are ambiguous. For example, volatility directly increases with α and uncertainty, but decreases when information is weaker because each trade reveals less information. Similarly, the bid-ask spread and the price impact of a trade ($p_{t+1} - p_t$) increase with uncertainty, but decrease when information is weaker.

[37]If I instead take the expectation of the absolute value, as in Malinova and Park (2010), because I consider only a single trade, the prediction is trivially always one. For this reason, I use a slightly different definition.

the probability that the first trader rushes, it is easy to show that

$$|E[IB]| = \mu_1 |2p_0 - 1| \left((2q_0 - 1)\beta_A^* + (2q_1 - 1)(1 - \beta_A^*)\right) \tag{6}$$

Simple comparative statics on (6) result in Prediction 1.

Prediction 1 (Order Imbalance): *The order imbalance decreases with an increase in (i) prior uncertainty about the asset value, and (ii) the probability of arrival of another trader.*

Prediction 1 states that a higher probability of trade arrival (which could perhaps be proxied with volume or media coverage) leads to more balanced order flows.[38] In contrast, by assumption, in the model with only a fixed opportunity cost of delay, competition has no effect. Easley et al. (1996) provide indirect evidence supporting the prediction of the competitive model. Because their probability of informed trading (PIN) measure increases with the order flow imbalance, a corollary of Prediction 1 is that it should decrease with prior uncertainty and competition. Easley et al. (1996) in fact find that higher volume is associated with a lower PIN.[39]

The prediction that the order imbalance decreases with an increase in uncertainty is driven by two reinforcing effects.[40] First, when uncertainty is higher, order flows are naturally more balanced ($p_0 = \frac{1}{2}$ results in an order imbalance of zero). Second, when traders rush more often, they have weaker information. The first effect is model independent, but the second effect is exactly opposite to that in the model with a monetary cost - higher uncertainty makes information more valuable so that traders pay more to acquire or process information in this case (Proposition 1). Because of the first effect, partialling out the second effect may be difficult without imposing structure in the empirical analysis (perhaps as in Cipriani and Guarino (2014), which I discuss below).

The relationship between the PIN measure of Easley et al. (1996) and uncertainty has

[38] Alternatively, as suggested by Malinova and Park (2010), changes in the probability trader arrival could result from index inclusion, deregulation, cross-listing, or international market openings.

[39] Barber and Odean (2008) find the opposite. They suggest that salient events attract individual traders which causes both volume and the order flow imbalance to increase because individual investors trade mostly in the same direction. Another traditional method of estimating the informational content of trades is to measure the persistence in prices using a VAR approach (Hasbrouck (1991)). However, the model's predictions with respect to price impact are ambiguous (see footnote 36).

[40] The model also predicts that deviations from fundamental values (mispricing) increase with uncertainty: $Pr(buy|V = 0) = Pr(sell|V = 1) = \mu_A\left(\beta_A^*(1 - q_0) + (1 - \beta_A^*)(1 - q_1)\right)$ increases with uncertainty. Limits to arbitrage (see Shleifer and Vishny (1997) and Mitchell et al. (2002)) can also cause mispricing when volatility is high due to increased arbitrage risk. To the extent volatility is related to uncertainty, these two explanations provide similar predictions. However, as discussed at the start of this section, volatility and uncertainty are not necessarily positively correlated.

been explored in cross-sectional studies (Kumar (2009) and Aslan et al. (2011)) with mixed results. In addition to the complication of the direct effect of uncertainty on the PIN, a second difficulty is in finding a proxy for uncertainty in the fundamental value of the asset that is uncorrelated with the information traders possess through any other channel.[41] To overcome this problem, a time-series study in which one can control for firm-specific effects may be better able to test the prediction. Potential proxies for uncertainty include the implied volatility index of the market (VIX), firm-specific implied volatility, or dispersion in analyst forecasts.[42]

The second observable for which the model makes predictions is the expected change in volume across periods. This expected change, which can be interpreted as a measure of the frequency of rushing, is given by

$$\begin{aligned} E[Vol_1 - Vol_0] &\equiv Pr(a_1 = buy) + Pr(a_1 = sell) - Pr(a_0 = buy) - Pr(a_0 = sell) \\ &= \mu_1(1 - 2\beta_A^*) \end{aligned} \tag{7}$$

Comparative statics on (7) result in Prediction 2.

Prediction 2 (Volume): *Volume shifts earlier in time due to an increase in (i) prior uncertainty about the asset value, (ii) the probability of arrival of another trader, and (iii) the first period signal quality.*

Prediction 2 states that announcements that generate more uncertainty should have higher concentrations of volume immediately after the announcement, relative to those that resolve uncertainty. Similarly, when it is more likely that other traders are processing the same information, volume should concentrate around the announcement. Finally, changes in regulations regarding disclosure or transparency, or improvements in technology that improve initial signal quality lead to similar concentration. For example, the internet and improvements in computing power have almost certainly improved the quality of initial information,

[41] Both Kumar (2009) and Aslan et al. (2011)) use proxies including firm age, firm size, monthly volume turnover, industry, and idiosyncratic volatility, and volatility in earnings. Each of these proxies may plausibly affect information through other channels or, as with volatility, is an output of the model that has no straightforward relationship to uncertainty.

[42] For other cross-sectional evidence that is suggestive of less informed trades when uncertainty is higher, the underreaction in stock prices is stronger in stocks with higher uncertainty (Zhang (2006) and Jiang et al. (2005)). One interpretation of underreaction, as summarized by Zhang (2006), is that underreaction is "more likely to reflect slow absorption of ambiguous information into stock prices than to reflect missing risk factors". Under this interpretation, the fact the prices of stocks with higher uncertainty more slowly absorb information is consistent with the model. However, strictly speaking, the model does not capture underreaction (prices follow a martingale). Extending the model to capture underreaction is an interesting avenue for future research.

while potentially leaving longer-term information quality unchanged, in which case the model predicts that volume should now be more concentrated around announcements than in the past.

The monetary cost model makes the same prediction with respect to initial signal quality, but the opposite prediction with respect to uncertainty. It predicts no change for an increase in the probability of trader arrival. On the other hand, models with exogenous information in which traders may time their trading decisions generate the same prediction with respect to the probability of trader arrival (for example, Holden and Subrahmanyam (1992) and Malinova and Park (2014)), but not with respect to a change in uncertainty or first period signal quality.[43] Therefore, the uncertainty prediction is unique to the competitive model.

The novel empirical predictions of the model are driven both by the time cost of acquiring information and the fact that poor quality information can result in misinformed traders. Recognizing this latter idea, Cipriani and Guarino (2014) relax the assumption of Easley at al. (1996) that informed traders have perfect information. They use the order flow imbalance to jointly identify the arrival rates of informed and uninformed traders in addition to the quality of information, finding that informed traders in fact have misinformation 40% of the time. Cipriani and Guarino (2014) analyze only a single stock using the entire available time series, but their methodology is well suited for testing the main mechanisms of the model. For example, one could estimate their model for a single stock at different points in time. The model predicts that low estimates of information quality coincide with high estimates for the rates of trader arrival. Or, one could divide time into periods of high and low uncertainty, perhaps based on the VIX, to test the comparative static predictions with respect to uncertainty. Future empirical work can combine the novel methodology of Cipriani and Guarino (2014) with the idea that traders may rush to trade in order to assess the quality of traders' information.

6 Conclusion

This paper considers a model in which information is acquired through time-consuming research. Adverse price movements due to others' trades impose a time cost of acquiring better information. This cost varies in a manner that causes traders to acquire the least information at high prior uncertainty, the same condition under which they acquire the most information at a monetary cost. The model therefore suggests that it is not without loss of generality to model information costs as monotonically increasing functions of signal quality, at least if one wants to interpret them as the reduced form of the cost of time.

[43] See also Admati and Pfleiderer (1988) and Foster and Viswanathan (1990).

The fact that information is acquired least when it contributes most to the long-term informational efficiency of the market (at high prior uncertainty) suggests that, if long-term information efficiency is socially desirable, we may want to reduce or eliminate the time cost, if possible. The model therefore relates to the debate on market design.[44] In markets that are continuously open, the fear of being preempted by other traders is present which, as shown, can cause traders to rush to trade on weak information. On the other hand, in markets that are cleared through call auctions, as long as the auctions are not too frequent relative to the time it takes to process information, traders can process information without fear. Thus, the model suggests that markets that are cleared through call auctions may be less prone to traders trading on weak information. Similarly, the model provides a potential justification for the practice of regulatory trading halts on stock exchanges when news is to be released, which ensure that traders have time to process information. Future theoretical work could extend the main idea of the model to explicitly address the market design question.

The model derives from the idea that private signals are generated by time-consuming research, in which case it is natural to assume that different researchers receive different signals. In some settings, however, different traders may have access to the same information through a news analytics service (for example, Ravenpack). It may be interesting in future work to consider how access to these types of services, which introduce competition in the information market, affect incentives to do time-consuming research.

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Appendix

Proofs

Preliminaries:
Throughout, I use the abbreviated notation:

$$Pr(a_t = buy|V = x) \equiv buy_{t|Vx}, x \in \{0,1\}, t \in \{0,1\}$$

$$Pr(a_t = sell|V = x) \equiv sell_{t|Vx}, x \in \{0,1\}, t \in \{0,1\}$$

$$Pr(e = e^i|V = x) \equiv e^i_{|Vx}, x \in \{0,1\}$$

To derive (2), apply Bayes' rule to (1). We have

$$\begin{aligned}
\pi_0(\beta, p_0, q_0) &= (p_0q_0 + (1-p_0)(1-q_0))\left(\tfrac{p_0q_0}{p_0q_0+(1-p_0)(1-q_0)} - \tfrac{p_0buy_{0|V=1})}{Pr(a_0=buy)}\right) \\
&+ (p_0(1-q_0) + (1-p_0)q_0)\left(\tfrac{p_0sell_{0|V=1})}{Pr(a_0=sell)} - \tfrac{p_0(1-q_0)}{p_0(1-q_0)+(1-p_0)q_0}\right) \\
&= \omega_0\left(\tfrac{q_0buy_{0|V=0}-(1-q_0)buy_{0|V=1})}{Pr(a_0=buy)}\right) \\
&+ \omega_0\left(\tfrac{q_0sell_{0|V=1}-(1-q_0)sell_{0|V=0}}{Pr(a_0=sell)}\right)
\end{aligned}$$

Symmetry implies $buy_{0|V=0} = sell_{0|V=1}$ and $buy_{0|V=1} = sell_{0|V=0}$ so that

$$\begin{aligned}
\pi_0(\beta, p_0, q_0) &= \omega_0\left(q_0buy_{0|V=0} - (1-q_0)buy_{1|V=1}\right) \\
&\times \left(\frac{1}{Pr(a_0 = buy)} + \frac{1}{Pr(a_0 = sell)}\right)
\end{aligned}$$

We have $buy_{0|V=0} = \mu(1-q_0)\beta + m$, where the first term is the probability of observing a purchase order from an informed trader (who is believed to rush with probability β) and the second term, $m = \frac{1-\mu}{4}$, is the probability of observing a purchase order at $t = 0$ from an uninformed trader. Similarly, $buy_{1|V=1} = \mu q_0\beta + m$. Substituting these expressions then gives (2).

The following expressions are used repeatedly and follow from straightforward algebra:

$$
\begin{aligned}
& Pr(a_t = buy, e = e^i) Pr(a_t = sell, e = e^i) \\
= \quad & Pr(e^i)^2 buy_{t|V1} buy_{t|V0} + \omega_0 e^i_{|V1} e^i_{|V0} \left(buy_{t|V1} - buy_{t|V0}\right)^2
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
& Pr(a_t = buy) Pr(a_t = sell) \\
= \quad & buy_{t|V1} buy_{t|V0} + \omega_0 \left(buy_{t|V1} - buy_{t|V0}\right)^2
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
& Pr(a_t = buy) Pr(a_t = sell) \\
= \quad & (2p_0 - 1)^2 buy_{t|V1} buy_{t|V0} + \omega_0 \left(buy_{t|V1} + buy_{t|V0}\right)^2
\end{aligned} \tag{10}
$$

The mathematical claim of Lemma A1 is used to prove several propositions.

Lemma A1: *The following inequality holds for any $x, y \in \mathbb{R}^+$, $n \geq 1$, and any $c_i, d_i \in [0,1]\ \forall i = 1 \ldots n$ satisfying $\sum_{i=1}^n c_i = \sum_{i=1}^n d_i = 1$ and at least one of c_i or d_i greater than zero $\forall i = 1 \ldots n$. Furthermore, it holds with equality if and only if $c_i = d_i \neq 0\ \forall i = 1 \ldots n$.*

$$
\sum_{i=1}^n \frac{c_i d_i}{c_i x + d_i y} \quad \leq \quad \frac{1}{x+y}
$$

Proof of Lemma A1: See the Online Appendix.□

Proof of Lemma 1:

The key to the proof is that the bid and ask prices are bounded by the possible beliefs the market maker may hold. Consider first a trader that has waited and received a positive signal, $s_1 = 1$. I claim that her private belief exceeds the maximum possible bid or ask price, so that buying is optimal. This belief is given by $Pr(V = 1|s_1 = 1) = \frac{p_0 q_1}{p_0 q_1 + (1-p_0)(1-q_1)}$ by Bayes' rule. The market maker's belief conditional on a buy is $Pr(V = 1|a_0 = \textit{no trade}, a_1 = buy)$, where no trade in the first period may reveal information if the informed trader rushes with positive probability and conditions her decision to delay her trade based upon the signal she receives. Regardless of the information acquisition and trading strategies of the informed trader, the highest possible belief the market maker can have, conditional on observing a buy decision, is

$$
\max A_1 \quad = \quad \frac{p_0(\mu q_1 + m)}{p_0(\mu q_1 + m) + (1-p_0)(\mu(1-q_1) + m)} \tag{11}
$$

This belief arises when only a trader with a positive, strong signal buys and is the highest possible because of two facts. First, the posterior is clearly higher if a buy order reveals a positive signal as opposed to a negative signal, or some combination of both (if, for example, those with strong, positive signals and those with weak, negative signals both buy). Second, conditional on revealing a positive signal, the belief is increasing in the quality of the signal.

Simple algebra shows that (11) is strictly smaller than the informed trader's belief for $\mu < 1$. The maximum possible bid price is the same as the maximum possible ask price, so is also smaller than the trader's belief. For a trader that has waited and received a negative signal, $s_1 = 0$, the argument reverses: her private belief must be below the minimum possible bid or ask price, so she must sell.

A similar argument establishes that a trader that has rushed and trades at $t = 0$ must buy with a positive signal and sell with a negative signal. In this case, her private belief with a positive signal is $Pr(V = 1|s_0 = 1) = \frac{p_0 q_0}{p_0 q_0 + (1-p_0)(1-q_0)}$ and the maximum ask price is the strictly smaller value given by

$$\max A_0 = \frac{p_0(\mu q_0 + m)}{p_0(\mu q_0 + m) + (1 - p_0)(\mu(1 - q_0) + m)}$$

Finally, consider the trading strategy of a trader who rushes but delays her trade until $t = 1$. In this case, the above argument no longer applies because the private belief of a trader with a positive signal is not necessarily larger than the maximum ask price in this period, given by (11). It is, however, larger than the maximum bid price, so selling is not optimal: given that a trader who waits to receive a strong signal sells when it is negative, the maximum possible bid price occurs when no trader waits and a trader who rushes but delays her trade buys with a negative signal:

$$\max B_1 = \frac{p_0(\mu q_0 + m)}{p_0(\mu q_0 + m) + (1 - p_0)(\mu(1 - q_0) + m)}$$

which is strictly less than the trader's private belief. Similarly, a trader who rushes and receives a negative signal can not buy at $t = 1$. A trader who rushes but delays her trading decision must then either not trade or trade in the direction of her private information. But, we can rule out delaying and not trading in equilibrium, because the trader can always instead trade in the direction of her signal at $t = 0$ and make a strictly positive expected profit.□

Proof of Lemma 2:

I first show that both $\pi_1(\beta, p_0, q_1)$ and $\pi_0(\beta, p_0, q_0)$ are symmetric with respect to $p_0 = \frac{1}{2}$, $\pi_i(\beta, p_0, q_1) = \pi_i(\beta, 1 - p_0, q_1)$, $i \in \{0, 1\}$. I provide the proof for $\pi_1(\beta, p_0, q_1)$. That for $\pi_0(\beta, p_0, q_0)$ is similar. From (4), we can write

$$\pi_1(\beta, p_0, q_1) = \omega_0 m(2q_1 - 1) \sum_{e^i \in E} \left(\frac{Pr(e = e^i|V = 0)Pr(e = e^i|V = 1)Pr(e^i)}{Pr(a_1 = buy, e = e^i)Pr(a_1 = sell, e = e^i)} \right)$$

Applying (8), we have

$$\pi_1(\beta, p_0, q_1) = \omega_0 m(2q_1 - 1) \sum_{e^i \in E} \left(\frac{Pr(e = e^i|V = 0)Pr(e = e^i|V = 1)Pr(e^i)}{Pr(e^i)^2 buy_{1|V1} buy_{1|V0} + \omega_0 e^i_{|V1} e^i_{|V0} \left(buy_{1|V1} - buy_{1|V0}\right)^2} \right)$$

Note that the prior only enters through $Pr(e^i)$ and ω_0. ω_0 is symmetric in p_0. $Pr(e^i)$ is not, but by the assumed symmetry of the set of possible public events, $Pr(e^i) = p_0 e^i_{|V1} + (1 - p_0)e^i_{|V0} = p_0 e^j_{|V0} + (1 - p_0)e^j_{|V1}$ for some event realization, e^j, and $p_0 e^j_{|V0} + (1 - p_0)e^j_{|V1}$

is the probability of event realization e^j in the expression for $\pi_1(\beta, 1-p_0, q_1)$. So, when one sums over all possible event realizations, $\pi_1(\beta, p_0, q_1) = \pi_1(\beta, 1-p_0, q_1)$.

To establish that the expected profits are concave and peak at $p_0 = \frac{1}{2}$, consider first $\pi_0(\beta, p_0, q_0)$. Take the derivative with respect to p_o:

$$\begin{aligned}\frac{\partial \pi_0(\beta, p_0, q_0)}{\partial p_0} &= (1-2p_0)m(2q_0-1)\left(\frac{1}{Pr(a_0=buy)}+\frac{1}{Pr(a_0=sell)}\right)\\ &- \omega_0 m(2q_0-1)\left(\frac{\frac{\partial Pr(a_0=buy)}{\partial p_0}}{Pr(a_0=buy)^2}+\frac{\frac{\partial Pr(a_0=sell)}{\partial p_0}}{Pr(a_0=sell)^2}\right)\end{aligned}$$

Now, $\frac{\partial Pr(a_0=buy)}{\partial p_0} = buy_{0|V1} - buy_{0|V0}$ and $\frac{\partial Pr(a_0=sell)}{\partial p_0} = sell_{0|V1} - sell_{0|V0} = buy_{0|V0} - buy_{0|V1}$ by symmetry so that simple algebra gives

$$\begin{aligned}&\frac{\partial \pi_0(\beta,p_0,q_0)}{\partial p_0}\\ =\ & (1-2p_0)m(2q_0-1)\frac{Pr(a_0=sell)+Pr(a_0=buy)}{Pr(a_0=buy)Pr(a_0=sell)}\left(1-\omega_0\frac{\left(buy_{0|V1}-buy_{0|V0}\right)^2}{Pr(a_0=buy)Pr(a_0=sell)}\right)\\ =\ & (1-2p_0)m(2q_0-1)\frac{buy_{0|V1}buy_{0|V0}(Pr(a_0=sell)+Pr(a_0=buy))}{Pr(a_0=buy)^2Pr(a_0=sell)^2}\end{aligned} \tag{12}$$

where the last step uses (9). This derivative is zero if and only if $p_0 = \frac{1}{2}$.

For the second derivative:

$$\begin{aligned}&\frac{\partial^2 \pi_0(\beta,p_0,q_0)}{\partial p_0^2}\\ =\ & -2m(2q_0-1)\left(Pr(a_0=sell)+Pr(a_0=buy)\right)\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)^2Pr(a_0=sell)^2}\\ -\ & (1-2p_0)m(2q_0-1)\left(Pr(a_0=sell)+Pr(a_0=buy)\right)\\ & \times buy_{0|V1}buy_{0|V0}\frac{2\left(Pr(a_0=buy)\frac{\partial Pr(a_0=sell)}{\partial p_0}+Pr(a_0=sell)\frac{\partial Pr(a_0=buy)}{\partial p_0}\right)}{Pr(a_0=buy)^3Pr(a_0=sell)^3}\end{aligned}$$

Canceling common terms, its sign is given by

$$\begin{aligned}& -1-(1-2p_0)\frac{\left(Pr(a_0=buy)\frac{\partial Pr(a_0=sell)}{\partial p_0}+Pr(a_0=sell)\frac{\partial Pr(a_0=buy)}{\partial p_0}\right)}{Pr(a_0=buy)Pr(a_0=sell)}\\ =\ & -1-(1-2p_0)(buy_{V0}-buy_{V1})\frac{(Pr(a_0=buy)-Pr(a_0=sell))}{Pr(a_0=buy)Pr(a_0=sell)}\\ =\ & -1-\frac{(1-2p_0)^2(buy_{0|V0}-buy_{0|V1})^2}{Pr(a_0=buy)Pr(a_0=sell)}\end{aligned}$$

which is always negative, so that the expected profit is everywhere concave.

To show concavity of $\pi_1(\beta, p_0, q_1)$, it is easier to use the expression

$$\pi_1(\beta, p_0, q_1) = \sum_{e^i \in E} Pr(e_i)\pi_1^{NP}(\beta, p_{e_i}, q_1) \tag{13}$$

directly. Taking the first and second derivatives with respect to the initial public belief, we obtain

$$
\begin{aligned}
\frac{\partial \pi_1(\beta, p_0, q_1)}{\partial p_0} &= \sum_{e^i \in E} \left\{ (e^i_{|V1} - e^i_{|V0}) \pi_1^{NP}(\beta, p_{e^i}, q_1) + Pr(e = e^i) \frac{\partial \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}} \frac{dp_{e^i}}{dp_0} \right\} \\
\frac{\partial^2 \pi_1(\beta, p_0, q_1)}{\partial p_0^2} &= \sum_{e^i \in E} \left\{ 2(e^i_{|V1} - e^i_{|V0}) \frac{\partial \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}} \frac{dp_{e^i}}{dp_0} \right. \\
&+ \left. Pr(e = e^i) \left(\frac{\partial^2 \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}^2} \left(\frac{dp_{e^i}}{dp_0} \right)^2 + \frac{\partial \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}} \frac{d^2 p_{e^i}}{dp_0^2} \right) \right\}
\end{aligned}
$$

We also have $p_{e^i} = Pr(V = 1|e^i) = \frac{p_0 e^i_{|V1}}{Pr(e^i)}$ by Bayes' rule. Its first and second derivatives are $\frac{dp_{e^i}}{dp_0} = \frac{e^i_{|V1} e^i_{|V0}}{Pr(e^i)^2}$ and $\frac{d^2 p_{e^i}}{dp_0^2} = -2 \frac{e^i_{|V1} e^i_{|V0}}{Pr(e^i)^3} (e^i_{|V1} - e^i_{|V0})$. Substituting these expressions and canceling common terms gives

$$
\frac{\partial^2 \pi_1(\beta, p_0, q_1)}{\partial p_0^2} = \sum_{e^i \in E} Pr(e = e^i) \frac{\partial^2 \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}^2} \left(\frac{e^i_{|V1} e^i_{|V0}}{Pr(e^i)^2} \right)^2
$$

In a manner identical to the proof for $\pi_0(\beta, p_0, q_0)$, one can show that $\pi_1^{NP}(\beta, p_0, q_1)$ is concave, which implies $\frac{\partial^2 \pi_1^{NP}(\beta, p_{e^i}, q_1)}{\partial p_{e^i}^2} < 0$. Therefore, $\frac{\partial^2 \pi_1(\beta, p_0, q_1)}{\partial p_0^2} < 0$ so that $\pi_1(\beta, p_0, q_1)$ is also concave. Finally, it is trivial to show that both $\pi_1(\beta, p_0, q_1)$ and $\pi_0(\beta, p_0, q_0)$ are zero at $p_0 = \{0, 1\}$.□

Proof of Proposition 1:

Part a). Consider first the decision of an informed trader who has rushed. Denote the probability with which she trades immediately as $\hat{\gamma}(s_0)$ (with corresponding belief of the market maker, $\gamma(s_0)$) which can depend upon the signal she receives. A trader who has a positive signal ($s_0 = 1$) prefers to trade at $t = 0$ if

$$
Pr(V = 1|s_0 = 1) - A_0 \geq Pr(V = 1|s_0 = 1) - A_1 - c
$$

From Lemma 1, only a trader with a positive signal buys at $t = 1$ so that, after observing a buy order at this time, observing no trade at $t = 0$ reveals no further information. Applying Bayes' rule, the above expression becomes

$$
\begin{aligned}
A_1 - A_0 &\geq -c \\
\iff \quad \frac{p_0 buy_{1|V1}}{p_0 buy_{1|V1} + (1-p_0) buy_{1|V0}} - \frac{p_0 buy_{0|V1}}{p_0 buy_{0|V1} + (1-p_0) buy_{0|V0}} &\geq -c
\end{aligned} \tag{14}
$$

A trader with a negative signal prefers to trade at $t = 0$ when

$$
\begin{aligned}
B_0 - B_1 &\geq -c \\
\iff \quad \frac{p_0 sell_{0|V1}}{p_0 sell_{0|V1} + (1-p_0) sell_{0|V0}} - \frac{p_0 sell_{1|V1}}{p_0 sell_{1|V1} + (1-p_0) sell_{1|V0}} &\geq -c
\end{aligned} \tag{15}
$$

In the limit as the probability of informed trading approaches zero, all trades are by uninformed traders so that the probability of observing a buy or a sell is $\frac{1}{4}$, independent of the time period and state. In this case, the left-hand side of each of (14) and (15) become zero so that, for a strictly positive cost, both hold with strict inequality. By continuity of the expressions, both must then also hold at a strictly positive value of μ, $\bar{\mu}_{11} > 0$. Therefore, for $\mu \leq \bar{\mu}_{11}$, we must have $\gamma^*(s_0 = 0) = \gamma^*(s_0 = 1) = 1$: a trader that rushes must trade immediately.

Given that the informed trader trades immediately after rushing, her expected profits from trading in periods 0 and 1 are given by (2) and (3), respectively. To establish uniqueness of the probability she rushes, β^{C*}, I show that $\pi_1^{NP}(\beta, p_0, q_1) - c - \pi_0(\beta, p_0, q_0)$ monotonically increases in β, by showing that $\pi_1^{NP}(\beta, p_0, q_1)$ monotonically increases in β, and that $\pi_0(\beta, p_0, q_0)$ monotonically decreases. For the former:

$$\frac{\partial \pi_1^{NP}(\beta, p_0, q_1)}{\partial \beta} = -\omega_0 m(2q_1 - 1)\left(\frac{\frac{\partial Pr(a_1=buy)}{\partial \beta}}{Pr(a_1 = buy)^2} + \frac{\frac{\partial Pr(a_1=sell)}{\partial \beta}}{Pr(a_1 = sell)^2}\right)$$

Evaluating the derivatives of the probabilities, we see that they are both negative. For brevity, I show only the derivative of the first probability:

$$\begin{aligned}\frac{\partial Pr(a_1=buy)}{\partial \beta} &= \frac{\partial(p_0(\mu q_1(1-\beta)+m)+(1-p_0)(\mu(1-q_1)(1-\beta)+m))}{\partial \beta} \\ &= -p_0\mu q_1 - (1-p_0)\mu(1-q_1)\end{aligned}$$

which is strictly less than zero.

$\frac{\partial \pi_0(\beta, p_0, q_0)}{\partial \beta}$ is evaluated similarly, but because the probabilities of buy and sell at $t = 0$ depend on β rather than $1 - \beta$, the sign of the derivative is reversed so that π_0 decreases in β.

Given that the gain in expected profits from waiting (net of costs) decreases in the probability that the informed trader waits, and that the informed trader's timing strategy and market maker's belief must coincide in equilibrium, there exists a unique fixed point, $\hat{\beta} = \beta \equiv \beta^{C*}$.

Part b). Assume $\mu \leq \bar{\mu}_{11}$ so that a unique equilibrium in which each trader trades in the period she acquires information exists (by part a)). Consider first the case in which the equilibrium probability of waiting, β^{C*}, is interior over some range of p_0 so that the implicit function theorem applies:

$$\frac{\partial \pi_1^{NP}(\beta^{C*}, p_0, q_1)}{\partial p_0} + \frac{\partial \pi_1^{NP}(\beta^{C*}, p_0, q_1)}{\partial \beta^{C*}}\frac{d\beta^{C*}}{dp_0} = \frac{\partial \pi_0(\beta^{C*}, p_0, q_0)}{\partial p_0} + \frac{\partial \pi_0(\beta^{C*}, p_0, q_0)}{\partial \beta^{C*}}\frac{d\beta^{C*}}{dp_0}$$

or

$$\frac{d\beta^{C*}}{dp_0} = \frac{\frac{\partial \pi_0(\beta^{C*},p_0,q_0)}{\partial p_0} - \frac{\partial \pi_1^{NP}(\beta^{C*},p_0,q_1)}{\partial p_0}}{\frac{\partial \pi_1^{NP}(\beta^{C*},p_0,q_1)}{\partial \beta^{C*}} - \frac{\partial \pi_0(\beta^{C*},p_0,q_0)}{\partial \beta^{C*}}} \tag{16}$$

From the proof of part a), the denominator of (16) is strictly positive. Using the expression for the first derivative with respect to p_0 from the proof of Lemma 2, (12), the numerator of (16) becomes

$$\begin{aligned}
& \tfrac{\partial \pi_0(\beta^{C*},p_0,q_0)}{\partial p_0} - \tfrac{\partial \pi_1^{NP}(\beta^{C*},p_0,q_1)}{\partial p_0} \\
= \quad & (1-2p_0)m\Big[(2q_0-1)\left(Pr(a_0=sell)+Pr(a_0=buy)\right)\tfrac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)^2 Pr(a_0=sell)^2} \\
& -(2q_1-1)\left(Pr(a_1=sell)+Pr(a_1=buy)\right)\tfrac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)^2 Pr(a_1=sell)^2}\Big] \\
= \quad & \tfrac{(1-2p_0)}{\omega_0}\Big[\pi_0(\beta^{C*},p_0,q_0)\tfrac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} \\
& -\pi_1^{NP}(\beta^{C*},p_0,q_1)\tfrac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)}\Big] \\
= \quad & \tfrac{(1-2p_0)}{\omega_0}\Big[\pi_0(\beta^{C*},p_0,q_0)\left(\tfrac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} - \tfrac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)}\right) \\
& -c\tfrac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)}\Big]
\end{aligned} \tag{17}$$

where the second equality uses the expressions for the expected profits, (2) and (3), and the last inequality uses the fact that we must have $\pi_1^{NP}(\beta^{C*},p_0,q_1)-c=\pi_0(\beta^{C*},p_0,q_0)$ in an equilibrium with β^{C*} interior.

In the limit as the probability of informed trading goes to zero, all trades are uninformed so that $\lim_{\mu\to 0}\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} = \lim_{\mu\to 0}\frac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)} = 1$. Then,

$$\begin{aligned}
& \lim_{\mu\to 0}\tfrac{\partial \pi_0(\beta^{C*},p_0,q_0)}{\partial p_0} - \tfrac{\partial \pi_1^{NP}(\beta^{C*},p_0,q_1)}{\partial p_0} \\
= \quad & -c\tfrac{(1-2p_0)}{\omega_0}
\end{aligned}$$

Therefore, in the limit, the sign of the numerator of (16) depends upon the sign of $(2p_0-1)$ so that $\frac{d\beta^{C*}}{dp_0}<0$ for $p_0<\frac{1}{2}$, $\frac{d\beta^{C*}}{dp_0}=0$ for $p_0=\frac{1}{2}$, and $\frac{d\beta^{C*}}{dp_0}>0$ for $p_0>\frac{1}{2}$. Because the bracketed term in (17) is strictly negative in the limit, by continuity, there must exist a strictly positive value of μ, $\bar{\mu}_{12}>0$ such that it remains negative for all $\mu\leq\bar{\mu}_{12}$, ensuring that the sign of the numerator of (16) continues to depend upon the sign of $(2p_0-1)$ and $\frac{d\beta^{C*}}{dp_0}$ reaches a minimum at $p_0=\frac{1}{2}$, provided β^{C*} remains interior.

As p_0 increases from one half, if β^{C*} reaches one at some $\hat{p}_0\geq\frac{1}{2}$, then $\beta^{C*}=1$ is the unique (by part a)) equilibrium for all $p_0\geq\hat{p}_0$. To see this fact, consider the difference in the expected profits for $p_0\geq\hat{p}_0$ as $\mu\to 0$:

$$\begin{aligned}
& \lim_{\mu\to 0}\pi_1(\beta,p_0,q_1)-c-\pi_0(\beta,p_0,q_0) \\
= \quad & 2\omega_0\left[(2q_1-1)-(2q_0-1)\right]-c
\end{aligned} \tag{18}$$

using the fact that all traders become uninformed. As p_0 increases from $\hat{p}_0\geq\frac{1}{2}$, (18) strictly decreases. Therefore, in the limit, if $\pi_1(0,\hat{p}_0,q_1)-c-\pi_0(0,\hat{p}_0,q_0)=0$ so that $\beta^{C*}=1$ at $\hat{p}_0$, then the difference in profits is strictly less than zero for all $p_0>\hat{p}_0$ so that $\beta^{C*}=1$ remains an equilibrium. Because the difference in profits is strictly negative in the limit, continuity ensures that there must exist a strictly positive value of μ, $\bar{\mu}_{13}$ such that it remains negative for all $\mu\leq\bar{\mu}_{13}$. By the same reasoning, if β^{C*} reaches zero at $\hat{p}_0>\frac{1}{2}$ as we decrease p_0 from one, then $\beta^{C*}=0$ is the unique equilibrium over $\frac{1}{2}\leq p_0\leq\hat{p}_0$. The corresponding statements for $p_0<\frac{1}{2}$ follow by symmetry around $p_0=\frac{1}{2}$.

Finally, set $\bar{\mu}_1 = \min\{\bar{\mu}_{11}, \bar{\mu}_{12}, \bar{\mu}_{13}\}$. Then, for $\bar{\mu}_1$ strictly positive, for all $\mu \leq \bar{\mu}_1$, the results of both part a) and part b) hold.$\square$

Proof of Lemma 3:

$\pi_1(\beta, p_0, q_1) < \pi_1^{NP}(\beta, p_0, q_1)$ is equivalent to

$$\sum_{e^i \in E} \left(\frac{e^i_{|V0} e^i_{|V1}}{Pr(a_1 = buy, e = e^i)} + \frac{e^i_{|V0} e^i_{|V1}}{Pr(a_1 = sell, e = e^i)} \right) \;<\; \left(\frac{1}{Pr(a_1 = buy)} + \frac{1}{Pr(a_1 = sell)} \right)$$

where the factor, $\omega_0 m(2q_1 - 1)$, cancels on either side of the inequality for $p_0 \in (0,1)$. To see that this inequality holds, apply the mathematical claim of Lemma A1 to the buy and sell terms separately. For the buy term, $Pr(a_1 = buy, e = e^i) = p_0 buy_{1|V1} e^i_{|V1} + (1-p_0) buy_{1|V0} e^i_{|V0}$ so set $c_i = e^i_{|V1}$,$d_i = e^i_{|V0}$, $x = p_0 buy_{1|V1}$, and $y = (1-p_0) buy_{1|V0}$ to apply the claim, and similarly for the sell term. We also know from the mathematical claim that as long as the event is informative, the inequality is strict.$\square$

Proof of Proposition 2:

Part a). As in the proof of Proposition 1, denote the equilibrium probability that the informed trader trades immediately after rushing as $\gamma^*(s_0)$. Consider first the informed trader's decision conditional on rushing and receiving a positive signal ($s_0 = 1$). She prefers to trade at $t = 0$ if

$$\begin{aligned}
& Pr(V=1|s_0=1) - A_0 \;\geq\; \textstyle\sum_{e^i \in E} Pr(e=e^i)\left(Pr(V=1|s_0=1,e^i)\right. \\
& \qquad\qquad \left. - Pr(V=1|a_1=buy, a_0=no\,trade, e^i)\right) \\
\iff\; & \tfrac{p_0 q_0}{Pr(s_0=1)} - \tfrac{p_0 buy_{0|V1}}{Pr(a_0=buy)} \;\geq\; \textstyle\sum_{e^i \in E}\Big[Pr(e=e^i) \\
& \qquad\qquad \times\left(\tfrac{p_0 q_0 e^i_{|V1}}{Pr(s_0=1,e=e^i)} - \tfrac{p_0 buy_{1|V1} e^i_{|V1}}{Pr(a_1=buy,e=e^i)}\right)\Big] \qquad (19) \\
\iff\; & \tfrac{q_0 buy_{0|V0} - (1-q_0) buy_{0|V1}}{Pr(s_0=1)Pr(a_0=buy)} \;\geq\; \textstyle\sum_{e^i \in E}\Big[e^i_{|V1} e^i_{|V0} Pr(e=e^i) \\
& \qquad\qquad \times \tfrac{\left(q_0 buy_{1|V0} - (1-q_0) buy_{1|V1}\right)}{Pr(s_0=1)Pr(e=e^i)Pr(a_1=buy,e=e^i)}\Big]
\end{aligned}$$

where the second equivalence uses the independence of s_0 and the public event. As $\mu \to 0$ so that $Pr(a_0 = buy) = \frac{1}{4}$ and $Pr(a_1 = buy, e = e^i) = \frac{1}{4} Pr(e = e^i)$ (all trades are uninformative), the inequality becomes

$$(2q_0 - 1) \;\geq\; \sum_{e^i \in E} \frac{e^i_{|V1} e^i_{|V0} (2q_0 - 1)}{Pr(e = e^i)}$$

The mathematical claim of Lemma A1 shows that, as long as one of the realizations of the public event is informative, $\sum_{e^i \in E} \frac{e^i_{|V1} e^i_{|V0}}{Pr(e=e^i)} < 1$, so that the informed trader has a strict incentive to trade at $t = 0$ (set $c_i = e^i_{|V1}$, $d_i = e^i_{|V0}$, $x = p_0$, and $y = 1 - p_0$ to apply the claim). Because both sides of (19) are continuous in μ and the inequality is strict in the limit, there exists a $\bar{\mu}_{21} > 0$ such that for all $\mu \leq \bar{\mu}_{21}$, the informed trader also prefers to trade at $t = 0$. Thus, for μ sufficiently small, $\gamma^*(s_0 = 1) = 1$ in any equilibrium. A similar calculation shows $\gamma^*(s_0 = 0) = 1$ is necessary as well.

It remains to show uniqueness of β^*. In the proof of Proposition 1, I show that $\frac{\partial \pi_0(\beta, p_0, q_0)}{\partial \beta} < 0$ and $\frac{\partial \pi_1^{NP}(\beta, p_0, q_1)}{\partial \beta} > 0$. Writing $\pi_1(\beta, p_0, q_1) = \sum_{e^i \in E} Pr(e = e^i)\pi_1^{NP}(\beta, p_{e^i}, q_1)$, it is clear that $\frac{\partial \pi_1(\beta, p_0, q_1)}{\partial \beta} > 0$, given that $\frac{\partial \pi_1^{NP}(\beta, p_0, q_1)}{\partial \beta} > 0$. Therefore, as in the proof of Proposition 1, the equilibrium probability of rushing, β^*, is unique.

Part b). Assume $\mu \leq \bar{\mu}_{21}$ so that a unique equilibrium in which each trader trades in the period she acquires information exists (by part a)). Consider first the case in which the equilibrium probability of waiting, β^*, is interior over some range of p_0. Applying the implicit function theorem as in Proposition 1, we obtain:

$$\frac{d\beta^*}{dp_0} = \frac{\frac{\partial \pi_0(\beta^*, p_0, q_0)}{\partial p_0} - \frac{\partial \pi_1(\beta^*, p_0, q_1)}{\partial p_0}}{\frac{\partial \pi_1(\beta^*, p_0, q_1)}{\partial \beta^*} - \frac{\partial \pi_0(\beta^*, p_0, q_0)}{\partial \beta^*}} \tag{20}$$

From the proof of Proposition 1, the denominator of (20) is strictly positive and for the numerator, we have

$$\frac{\partial \pi_0(\beta^*, p_0, q_0)}{\partial p_0} = \frac{(1 - 2p_0)}{\omega_0}\pi_0(\beta^*, p_0, q_0)\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0 = buy)Pr(a_0 = sell)}$$

Taking the derivative of (4) with respect to p_0 after combining the buy and sell terms in the summation, we get

$$\frac{\partial \pi_1(\beta^*, p_0, q_1)}{\partial p_0} = \frac{(1 - 2p_0)}{\omega_0}\pi_1(\beta^*, p_0, q_1) + \Sigma$$

where

$$\begin{aligned}\Sigma \equiv\ & \omega_0 m(2q_1 - 1)(buy_{1|V0} + buy_{1|V1}) \\ & \times \sum_{e^i \in E} e^i_{|V=0} e^i_{|V=1} \left[\frac{(e^i_{|V=1} - e^i_{|V=0})Pr(a_1 = buy, e = e^i)Pr(a_1 = sell, e = e^i)}{Pr(a_1 = buy, e = e^i)^2 Pr(a_1 = sell, e = e^i)^2} \right. \\ & \left. - \frac{Pr(e = e^i)\frac{\partial}{\partial p_0}\left(Pr(a_1 = buy, e = e^i)Pr(a_1 = sell, e = e^i)\right)}{Pr(a_1 = buy, e = e^i)^2 Pr(a_1 = sell, e = e^i)^2} \right]\end{aligned}$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial p_0}\left(Pr(a_1=buy,e=e^i)Pr(a_1=sell,e=e^i)\right)\\
=\;&\frac{\partial}{\partial p_0}\left(Pr(e^i)^2 buy_{1|V1}buy_{1|V0}+\omega_0 e^i_{|V1}e^i_{|V0}\left(buy_{1|V1}-buy_{1|V0}\right)^2\right)\\
=\;&2Pr(e^i)(e^i_{|V=1}-e^i_{|V=0})buy_{1|V1}buy_{1|V0}+(1-2p_0)e^i_{|V1}e^i_{|V0}\left(buy_{1|V1}-buy_{1|V0}\right)^2
\end{aligned}
$$

where the first equivalence uses (8). The numerator of (20) is then

$$
\begin{aligned}
&\frac{\partial\pi_0(\beta^*,p_0,q_0)}{\partial p_0}-\frac{\partial\pi_1(\beta^*,p_0,q_1)}{\partial p_0}\\
=\;&\frac{(1-2p_0)}{\omega_0}\pi_0(\beta^*,p_0,q_0)\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)}-\frac{(1-2p_0)}{\omega_0}\pi_1(\beta^*,p_0,q_1)-\Sigma
\end{aligned}
$$

When β^* is interior, $\pi_0(\beta^*,p_0,q_0)=\pi_1(\beta^*,p_0,q_1)$, so that this expression becomes

$$
\begin{aligned}
&\frac{\partial\pi_0(\beta^*,p_0,q_0)}{\partial p_0}-\frac{\partial\pi_1(\beta^*,p_0,q_1)}{\partial p_0}\\
=\;&\frac{(1-2p_0)}{\omega_0}\pi_0(\beta^*,p_0,q_0)\left[\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)}-1\right]-\Sigma
\end{aligned}
\tag{21}
$$

In the limit as $\mu\to 0$, the first term approaches zero as in Proposition 1. The sign of $\frac{d\beta^*}{dp_0}$ is then determined by

$$
\begin{aligned}
&-\lim_{\mu\to 0}\Sigma\\
=\;&-\frac{\omega_0(2q_1-1)}{8}\sum_{e^i\in E}e^i_{|V=0}e^i_{|V=1}\left[\frac{16(e^i_{|V=1}-e^i_{|V=0})}{Pr(e=e^i)^2}-\frac{32(e^i_{|V=1}-e^i_{|V=0})}{Pr(e=e^i)^2}\right]\\
=\;&\omega_0 2(2q_1-1)\sum_{e^i\in E}e^i_{|V=0}e^i_{|V=1}\frac{(e^i_{|V=1}-e^i_{|V=0})}{Pr(e=e^i)^2}
\end{aligned}
$$

From Lemma A2 of the Online Appendix, $\sum_{e^i\in E}e^i_{|V=0}e^i_{|V=1}\frac{(e^i_{|V=1}-e^i_{|V=0})}{Pr(e=e^i)^2}$ has the same sign as $(1-2p_0)$, so that, in the limit, $\frac{d\beta^*}{dp_0}$ is strictly increasing for $p_0<\frac{1}{2}$, zero at $p_0=\frac{1}{2}$, and strictly decreasing for $p_0>\frac{1}{2}$. By continuity, because these properties of $\frac{d\beta^*}{dp_0}$ are strict, there exists a $\bar{\mu}_{22}>0$ such that for all $\mu\leq\bar{\mu}_{22}$, $\frac{d\beta^*}{dp_0}$ peaks at $p_0=\frac{1}{2}$.

As p_0 increases from one half, if β^* reaches zero at some $\hat{p}_0\geq\frac{1}{2}$, then $\beta^*=0$ is the unique (by part a)) equilibrium for all $p_0\geq\hat{p}_0$.To see this fact, consider the relationship between the expected profits for $p_0\geq\hat{p}_0$ as $\mu\to 0$:

$$
\begin{aligned}
&\lim_{\mu\to 0}\pi_1(\beta,p_0,q_1)-\pi_0(\beta,p_0,q_0)\\
=\;&2\omega_0\left[(2q_1-1)\sum_{e^i\in E}\frac{e^i_{|V=0}e^i_{|V=1}}{Pr(e=e^i)}-(2q_0-1)\right]
\end{aligned}
\tag{22}
$$

using the fact that all traders become uninformed. The sign of (22) is determined by the term in brackets which is strictly increasing for $p_0>\frac{1}{2}$ because

$$
\begin{aligned}
&\frac{\partial}{\partial p_0}\sum_{e^i\in E}\left[(2q_1-1)\sum_{e^i\in E}\frac{e^i_{|V=0}e^i_{|V=1}}{Pr(e=e^i)}-(2q_0-1)\right]\\
=\;&-(2q_1-1)\sum_{e^i\in E}e^i_{|V=0}e^i_{|V=1}\frac{(e^i_{|V=1}-e^i_{|V=0})}{Pr(e=e^i)^2}
\end{aligned}
$$

which, by Lemma A2, is strictly greater than zero over this range. Therefore, in the limit, if $\pi_1(0,\hat{p}_0,q_1)-\pi_0(0,\hat{p}_0,q_0)=0$ so that $\beta^*=0$ at $\hat{p}_0$, then the difference in profits is strictly greater than zero for all $p_0 > \hat{p}_0$ so that $\beta^* = 0$ remains an equilibrium. Because the difference in profits is strictly positive in the limit, continuity ensures that there must exist a strictly positive value of μ, $\bar{\mu}_{23}$ such that it remains negative for all $\mu \leq \bar{\mu}_{23}$. By the same reasoning, if β^* reaches one at $\hat{p}_0 > \frac{1}{2}$ as we decrease p_0 from one, then $\beta^* = 1$ is the unique equilibrium over $\frac{1}{2} \leq p_0 \leq \hat{p}_0$. The corresponding statements for $p_0 < \frac{1}{2}$ follow by symmetry around $p_0 = \frac{1}{2}$.

Finally, set $\bar{\mu}_2 = \min\{\bar{\mu}_{21}, \bar{\mu}_{22}, \bar{\mu}_{23}\}$. Then, for $\bar{\mu}_2$ strictly positive, for all $\mu \leq \bar{\mu}_2$, the results of both part a) and part b) hold.□

Proof of Proposition 3:

Assume $\mu \leq \bar{\mu}_2$ so that a unique equilibrium in which each trader trades in the period she acquires information exists (by Proposition 2). With $\beta^* \in (0,1)$ so that the monetary cost, $c^P(p_0, q_0, q_1)$, is uniquely defined, by the implicit function theorem, we have

$$\frac{\partial c^P(p_0, q_0, q_1)}{\partial \beta^*} = \left(\frac{\partial \pi_1^{NP}(\beta^*, p_0, q_1)}{\beta^*} - \frac{\partial \pi_0(\beta^*, p_0, q_0)}{\beta^*} \right) \frac{\partial \beta^*}{\partial p_0}$$

From the proof of Proposition 1, the term in parentheses is strictly positive so that $\frac{\partial c^P(p_0,q_0,q_1)}{\partial \beta^*}$ has the same sign as $\frac{\partial \beta^*}{\partial p_0}$. But, from Proposition 2, we know that, provided it is interior, β^* strictly increases when $p_0 < \frac{1}{2}$, reaches a maximum at $p_0 = \frac{1}{2}$, and then strictly decreases for $p_0 > \frac{1}{2}$. Therefore, $c^P(p_0, q_0, q_1)$ does as well.□

Proof of Proposition 4:

Part a). I first establish properties of the equilibrium strategy, β^{NP*}, for the case in which trader B knows she faces no intervening trade because trader A already traded. The claim is that there is a unique equilibrium in this case in which trader B rushes with probability $\beta^{NP*} \in [0, \frac{1}{2})$ and trades in the period she acquires information. Consider first the claim that, when trader B rushes, she trades immediately in any equilibrium. As in the proof of Proposition 1, denote the probability that the informed trader trades immediately after rushing as $\hat{\gamma}(s_0)$ (with corresponding belief of the market marker, $\gamma(s_0)$). Unlike in that proof, in the absence of a public event, there is no strict incentive to trade immediately as $\mu_B \to 0$. However, the following argument (I omit the mathematical details, but they are available upon request) shows that, in the absence of a public event, the informed trader trades immediately after rushing for all $\mu_B > 0$.

In the absence of a public event (only), both types of informed traders (based on their signal realization after rushing), must follow the same strategy of delaying trade or not, $\hat{\gamma}(s_0 = 0) = \hat{\gamma}(s_0 = 1)$. Given this fact, if both types were to delay trade with any positive probability, $\hat{\gamma}(s_0 = 0) = \hat{\gamma}(s_0 = 1) < 1$, there is a profitable deviation. Any time the informed trader rushes and then delays trade, she can instead wait to obtain the better signal. Doing so earns a strictly higher profit because, the second period bid and ask prices are unchanged, but the better quality private information leads to strictly higher profits.

Given that trader B trades in the period she acquires information, it is easily established that β^{NP*} is unique: it is a special case of Proposition 1 (with $c = 0$). Finally, I omit the proof that $\beta^{NP*} \in [0, \frac{1}{2})$, but the proof is available upon request. Intuitively, with equal signal strengths, the symmetry of the problem requires $\beta^{NP*} = \frac{1}{2}$, and in the absence of any cost, a stronger second period signal induces the informed trader to rush with lower probability.

Now, consider the case in which either trader faces an intervening trade by the other. Given that intervening trades are informative and that the proof of Proposition 2 was for any informative public event, then, provided $\mu_A \leq \bar{\mu}_2$ and $\mu_B \leq \bar{\mu}_2$, each trader must trade immediately if she rushes. It remains to show that β_A^* and β_B^* (when trader A has not traded) are unique.

From the proof of Proposition 2, each $\pi_{i,1}(\beta_A^*, \beta_B^*, p_0, q_1) - \pi_0(\beta_i^*, p_0, q_0)$ is strictly monotonic in β_i^* which ensures $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) - \pi_0(\beta_A^*, p_0, q_0)$ is also strictly monotonic in β_A^*.[45] Therefore, each trader has a unique best response to the timing strategy of the other trader. I now show that β_A^* is weakly increasing in β_B^*, and β_B^* is weakly decreasing in β_A^* which, together with the uniqueness of best responses, ensures a unique fixed point.

Consider trader A. I show that her expected profit from waiting, $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) = \alpha\pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) + (1-\alpha)\pi_1^{NP}(\beta_A^*, p_0, q_1)$, is decreasing in β_B^*. The second term is independent of trader B's strategy. The first term depends upon trader B's strategy through the summation over the set of public events corresponding to the possibility of her trade: $e^1 = Pr(buy)$, $e^2 = Pr(sell)$, and $e^3 = Pr(no\ trade)$. I first show that the summation over events when trader A buys is decreasing in β_B^*. Expanding this term:

$$\begin{aligned}
& \textstyle\sum_{e^i \in E} \frac{Pr(e^i|V=0)Pr(e^i|V=1)}{Pr(a_{A,1}=buy, e=e^i)} \\
= \;& \frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{A,1}=buy, e=e^1)} + \frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{A,1}=buy, e=e^2)} + \frac{e^3_{|V0}}{Pr(a_{A,1}=buy)} \\
= \;& Pr(a_{A,1}=buy)\left(\frac{e^1_{|V0}e^1_{|V1}\left(\mu_B\beta_B^*+2m_B\right)}{Pr(a_{A,1}=buy, e=e^1)Pr(a_{A,1}=buy, e=e^2)} + \frac{e^3_{|V0}}{Pr(a_{A,1}=buy)^2}\right)
\end{aligned}$$

where the first equality uses symmetry: $e^1_{|V0} = e^2_{|V1} = \mu_B\beta_B^*(1-q_0) + m_B$, $e^1_{|V1} = e^2_{|V0} = \mu_B\beta_B^* q_0 + m_B$, and $e^3_{|V0} = e^3_{|V1} = \mu_B(1-\beta_B^*) + 2m_B$. Dividing by $Pr(a_{A,1} = buy)$, which is independent of β_B^*, and taking the derivative with respect to β_B^*:

$$\begin{aligned}
& \frac{1}{Pr(a_{A,1}=buy)}\frac{\partial}{\partial\beta_B^*}\textstyle\sum_{e^i \in E}\frac{Pr(e^i|V=0)Pr(e^i|V=1)}{Pr(a_{A,1}=buy, e=e^i)} \\
= \;& \mu_B\left(\frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{A,1}=buy, e=e^1)Pr(a_{A,1}=buy, e=e^2)} - \frac{1}{Pr(a_{A,1}=buy)^2}\right) \\
& + \left(\mu_B\beta_B^* + 2m_B\right)\frac{\partial}{\partial\beta_B^*}\frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{A,1}=buy, e=e^1)Pr(a_{A,1}=buy, e=e^2)}
\end{aligned} \tag{23}$$

Each of the two terms in (23) is separately negative. Expanding the product of probabilities gives $Pr(a_{A,1} = buy, e = e^1)Pr(a_{A,1} = buy, e = e^2) = e^1_{|V0}e^1_{|V1}Pr(a_{A,1} = buy)^2 + buy_{|V0}buy_{|V1}\omega_0(e^1_{|V1} - e^1_{|V0})^2$. Cross-multiplying the first term and substituting this expression, its sign becomes that of $Pr(a_{A,1} = buy)^2 e^1_{|V0}e^1_{|V1} - e^1_{|V0}e^1_{|V1}Pr(a_{A,1} = buy)^2 - buy_{|V0}buy_{|V1}\omega_0(e^1_{|V1} - e^1_{|V0})^2 = -buy_{|V0}buy_{|V1}\omega_0(e^1_{|V1} - e^1_{|V0})^2$, which is weakly negative.

For the second term, to evaluate the derivative it is convenient to first multiply by the term $Pr(a_{A,1} = buy)^2$, which is independent of β_B^* . Doing so, we see that the denominator of the derivative is positive and the sign of the numerator is determined by

[45] The additional term in the denominator of $\pi_{B,1}(\beta_A^*, \beta_B^*, p_0, q_1)$ is independent of β_B^* so does not affect this monotonicity result.

$$
\begin{aligned}
& Pr(a_{A,1}=buy)^2 \frac{\partial}{\partial \beta_B^*}\left(e^1_{|V0}e^1_{|V1}\right)\left(e^1_{|V0}e^1_{|V1}Pr(a_{A,1}=buy)^2 + buy_{|V0}buy_{|V1}\omega_0(e^1_{|V1}-e^1_{|V0})^2\right) \\
& -Pr(a_{A,1}=buy)^2 e^1_{|V0}e^1_{|V1}\left(\frac{\partial}{\partial \beta_B^*}\left(e^1_{|V0}e^1_{|V1}\right)Pr(a_{A,1}=buy)^2 + buy_{|V0}buy_{|V1}\omega_0\frac{\partial}{\partial \beta_B^*}(e^1_{|V1}-e^1_{|V0})^2\right) \\
= \; & Pr(a_{A,1}=buy)^2 buy_{|V0}buy_{|V1}\omega_0\left(\frac{\partial}{\partial \beta_B^*}\left(e^1_{|V0}e^1_{|V1}\right)(e^1_{|V1}-e^1_{|V0})^2\right. \\
& \left. -2e^1_{|V0}e^1_{|V1}(e^1_{|V1}-e^1_{|V0})\frac{\partial}{\partial \beta_B^*}(e^1_{|V1}-e^1_{|V0})\right)
\end{aligned}
$$

which has a sign determined by

$$
\begin{aligned}
& (e^1_{|V1}-e^1_{|V0})\frac{\partial}{\partial \beta_B^*}\left(e^1_{|V0}e^1_{|V1}\right)(e^1_{|V1}-e^1_{|V0}) - 2e^1_{|V0}e^1_{|V1}\frac{\partial}{\partial \beta_B^*}(e^1_{|V1}-e^1_{|V0}) \\
= \; & (e^1_{|V1}-e^1_{|V0})\left((\mu_B(1-q_0)e^1_{|V1}+\mu_B q_0 e^1_{|V0})\mu_B\beta_B^*(2q_0-1) - 2\mu_B e^1_{|V0}e^1_{|V1}(2q_0-1)\right) \\
= \; & (e^1_{|V1}-e^1_{|V0})(2q_0-1)\mu_B\left(\mu_B\beta_B^*((1-q_0)e^1_{|V1}+q_0e^1_{|V0}) - 2e^1_{|V0}e^1_{|V1}\right) \\
= \; & (e^1_{|V1}-e^1_{|V0})(2q_0-1)\mu_B\left(\mu_B\beta_B^*((1-q_0)(\mu_B\beta_B^*q_0+m_B)\right. \\
& \left. +q_0(\mu_B\beta_B^*(1-q_0)+m_B) - 2(\mu_B\beta_B^*q_0+m_B)(\mu_B\beta_B^*(1-q_0)+m_B)\right) \\
= \; & (e^1_{|V1}-e^1_{|V0})(2q_0-1)\mu_B\left(-\mu_B m_B\beta_B^* - 2m_B\right) \\
< \; & 0
\end{aligned}
$$

Similarly, the second term of $\pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ (corresponding to trader A selling) also strictly decreases in β_B^*, so that her overall expected profit does as well. The best response of trader A, $\beta_A^*(\beta_B^*)$ is governed by $\Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)-\pi_0(\beta_A^*,p_0,q_0)=0$ when β_A^* is interior. From the monotonicity results in the proofs of Propositions 1 and 2, if $\Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ strictly decreases in β_B^*, then we must have β_A^* strictly increase in β_B^* when interior. If, instead, $\beta_A^*\in\{0,1\}$ then it is unaffected by β_B^*.

Now consider trader B. Opposite to the case of trader A above, we can show that $\pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ is increasing in β_A^*. When trader A has not already traded, the set of public events corresponding to the trade of trader A is $e^1=Pr(buy)$ and $e^2=Pr(sell)$. The conditional probabilities are $e^1_{|V0}=e^2_{|V1}=\mu_A(1-\beta_A^*)(1-q_1)+m_A$ and $e^1_{|V1}=e^2_{|V0}=\mu_A(1-\beta_A^*)q_1+m_A$. The term in $\pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ corresponding to trader B buying is given by

$$
\begin{aligned}
& \frac{1}{\mu_A(1-\beta_A^*)+2m_A}\sum_{e^i\in E}\frac{Pr(e^i|V=0)Pr(e^i|V=1)}{Pr(a_{B,1}=buy,e=e^i)} \\
= \; & \frac{1}{\mu_A(1-\beta_A^*)+2m_A}\left(\frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{B,1}=buy,e=e^1)}+\frac{e^1_{|V0}e^1_{|V1}}{Pr(a_{B,1}=buy,e=e^2)}\right) \qquad (24) \\
= \; & \frac{e^1_{|V0}e^1_{|V1}Pr(a_{B,1}=buy)}{Pr(a_{B,1}=buy,e=e^1)Pr(a_{B,1}=buy,e=e^2)}
\end{aligned}
$$

because $Pr(a_{B,1}=buy,E=e^1)+Pr(a_{B,1}=buy,E=e^2)=Pr(a_{B,1}=buy)(Pr(e=e^1)+Pr(e=e^2))=Pr(a_{B,1}=buy)(\mu_A(1-\beta_A^*)+2m_A)$. If we divide (24) by $Pr(a_{B,1}=buy)$, which is independent of β_A^*, we can see that its derivative with respect to β_A^* is identical to the second term in (23) except that the public events depend upon $1-\beta_A^*$ instead of β_B^* so it is strictly positive instead of strictly negative. Similarly, the term in $\pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ corresponding to trader B selling, and therefore her expected profit from waiting overall, is strictly increasing in β_A^*. If $\pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)$ strictly increases in β_A^* then from the monotonicity properties of Propositions 1 and 2, β_B^* must strictly decrease in β_A^* if interior. If $\beta_A^* \in \{0,1\}$, it is unaffected by β_B^*. Because the two best response functions are weakly monotonic (strict when interior), and of opposite sign, the fixed point of best responses is guaranteed to be unique.$\square$

For the remaining parts, continue to assume $\mu_A \leq \bar{\mu}_2$ and $\mu_B \leq \bar{\mu}_2$ so that the unique equilibrium is one in which each trader trades in the period she acquires information (by part a)).

Part b). To see that we must have trader A rush more often than if there is no competition, suppose not: $\beta_A^* \leq \beta^{NP*}$. Because $\beta^{NP*} < \frac{1}{2}$, trader A waits with positive probability and her trade impacts trader B whenever she does. By part a), we must then have $\beta_B^* \geq \beta^{NP*}$ (with strict inequality if $\beta^{NP*} > 0$). This fact in turn implies trader B impacts trader A's expected profit from waiting so that $\beta_A^* \geq \beta^{NP*}$ (with strict inequality if $\beta^{NP*} > 0$). We then have a contradiction if $\beta^{NP*} > 0$. If $\beta^{NP*} = 0$, $\beta_A^* = \beta^{NP*} = 0$ is possible. That trader B rushes weakly more often in the presence of competition is immediate: if trader A rushes with probability one, then trader B unconditionally mixes with probability β^{NP*} and if trader A instead waits with positive probability, then on observing trader A wait, $\beta_B^* \geq \beta^{NP*}$ by part a).

Part c). Consider an increase in α. If $\beta_B^* = 0$, trader B has no impact on trader A so that clearly any change in α has no effect on trader A's probability of rushing. If $\beta_B^* > 0$, on the other hand, the direct effect of an increase in α on trader A's expected profit from waiting is given by $\frac{\partial \Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)}{\partial \alpha} = \pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) - \pi_1^{NP}(\beta_A^*, p_0, q_1)$ which is strictly negative due to the impact of trader B (see the proof of part a)).

When $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)$ decreases, both β_A^* and β_B^* may change provided $\beta_A^* \in (0,1)$ (if not, β_A^* may remain unchanged). After the change, I claim that β_A^* must be strictly larger so that $\frac{d\beta_A^*}{d\alpha} > 0$. Suppose not. If β_A^* remains unchanged, then β_B^* remains unchanged given that trader B's best response is unique (part a)), so we have $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) < \pi_0(\beta_A^*, p_0, q_0)$ contradicting $\beta_A^* \in (0,1)$ which requires the two expected profits to be equal. If β_A^* were to decrease, then β_B^* weakly increases as seen in the proof of part a). But, as also shown there, this increase in the probability that trader B rushes further decreases $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)$ so that again $\Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1) < \pi_0(\beta_A^*, p_0, q_0)$, contradicting $\beta_A^* \in (0,1)$. Therefore β_A^* doesn't change if either $\beta_B^* = 0$ or $\beta_A^* \in \{0,1\}$, but must otherwise strictly increase when α increases.

Part d). Consider first the case of β_A^* interior so that we can apply the implicit function theorem

$$\frac{d\beta_A^*}{dp_0} = \frac{\frac{\partial \pi_{A,0}(\beta_A^*, p_0, q_0)}{\partial p_0} - \frac{\partial \Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)}{\partial p_0} - \frac{\partial \Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)}{\partial \beta_B^*} \frac{d\beta_B^*}{dp_0}}{\frac{\partial \Pi_{A,1}(\beta_A^*, \beta_B^*, p_0, q_1)}{\partial \beta_1^*} - \frac{\partial \pi_{A,0}(\beta^*, p_0, q_0)}{\partial \beta_1^*}}$$

Comparing to the exogenous public event case, (20), we see that the additional equilibrium effect of a change in β_B^* enters the numerator. If β_A^* is interior, then either β_B^* is interior or $\beta_B^* = 1$.[46]

[46]If $\beta_B^* = 0$, trader A rushes with probability β^{NP*}. But then because we are considering the case of β_A^*

If $\beta_B^* = 1$, $\frac{d\beta_B^*}{dp_0} = 0$. From the proof of part a), $\frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_1^*} - \frac{\partial \pi_{A,0}(\beta^*,p_0,q_0)}{\partial \beta_1^*} > 0$ so that the sign of $\frac{d\beta_A^*}{dp_0}$ depends upon the sign of $\frac{\partial \pi_{A,0}(\beta_A^*,p_0,q_0)}{\partial p_0} - \frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial p_0}$. Exactly as in the proof of Proposition 2, we can relate the derivatives to their corresponding expected profits and then use the equilibrium relationship between expected profits to substitute out one of them out. Doing so, in the limit as $\mu_A \to 0$, we have

$$\begin{aligned} & \lim_{\mu_A \to 0} \frac{\partial \pi_{A,0}(\beta_A^*,p_0,q_0)}{\partial p_0} - \frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial p_0} \\ = \; & -\lim_{\mu_A \to 0} \alpha \Sigma \end{aligned}$$

As in the proof of Proposition 2, $-\lim_{\mu_A \to 0} \Sigma$ has the same sign as $(1-2p_0)$ provided at least one of the event realizations is informative, which is the case if $\beta_B^* = 1$. Therefore, in this case, $\frac{d\beta_A^*}{dp_0}$ has a strict maximum at $p_0 = \frac{1}{2}$, which, by continuity, ensures that there must exist a strictly positive value of μ_A, $\bar{\mu}_{31}$, such that it also has a strict maximum for $\mu_A \leq \bar{\mu}_{31}$.

If β_B^* is interior, we can substitute out $\frac{d\beta_B^*}{dp_0}$ by applying the implicit function theorem to the equilibrium relationship for β_B^*. Doing so, and solving for $\frac{d\beta_A^*}{dp_0}$, results in

$$\frac{d\beta_A^*}{dp_0} = \frac{\frac{\partial \pi_{A,0}(\beta_A^*,p_0,q_0)}{\partial p_0} - \frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial p_0} - A\left(\frac{\partial \pi_{B,0}(\beta_B^*,p_0,q_0)}{\partial p_0} - \frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial p_0}\right)}{\frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_1^*} - \frac{\partial \pi_{A,0}(\beta^*,p_0,q_0)}{\partial \beta_1^*} - A\frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_A^*}} \tag{25}$$

where

$$A \equiv \frac{\frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_2^*}}{\frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_2^*} - \frac{\partial \pi_{B,0}(\beta_B^*,p_0,q_0)}{\partial \beta_2^*}}$$

From the proof of part a), $\frac{\partial \Pi_{A,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_2^*} < 0$ and $\frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_2^*} - \frac{\partial \pi_{B,0}(\beta_B^*,p_0,q_0)}{\partial \beta_2^*} > 0$ so that $A < 0$. In addition, $\frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial \beta_A^*} > 0$, so that the denominator of (25) is strictly positive.

For the numerator, consider the limit as $\mu_A \to 0$. As in the $\beta_B^* = 1$ case, there exists a strictly positive value of μ_A, $\bar{\mu}_{32}$, such that the difference of the first two terms peaks at $p_0 = \frac{1}{2}$ for $\mu_A \leq \bar{\mu}_{32}$. For the last term, in the limit as $\mu_A \to 0$, trader A has no impact on trader B so that trader B rushes with probability $\beta_B^* = \beta^{NP*}$. In this case, it is easy to show that $\frac{\partial \pi_{B,0}(\beta_B^*,p_0,q_0)}{\partial p_0} - \frac{\partial \pi_{B,1}(\beta_A^*,\beta_B^*,p_0,q_1)}{\partial p_0}$ vanishes (see (17) with $c = 0$). Therefore, for $\mu_A \leq \bar{\mu}_{32}$, $\frac{d\beta_A^*}{dp_0}$ has a strict maximum at $p_0 = \frac{1}{2}$, provided β_A^* is interior.

Using the same argument as in the proof of Proposition 2, we can show that, as p_0 increases from one half, if β_A^* reaches zero at some $\hat{p}_0 \geq \frac{1}{2}$, then $\beta_A^* = 0$ is the unique (by part a)) equilibrium for all $p_0 \geq \hat{p}_0$. And, if β_A^* reaches one at $\hat{p}_0 > \frac{1}{2}$ as we decrease p_0 from one, then $\beta_A^* = 1$ is the unique equilibrium over $\frac{1}{2} \leq p_0 \leq \hat{p}_0$. The only difference from the public event case is that the relationship between the expected profits for $p_0 \geq \hat{p}_0$ as $\mu_A \to 0$ is given by

interior, β^{NP*} must be interior. This fact in turn implies trader A impacts trader B because trader A waits with positive probability, so we must have $\beta_B^* \geq \beta^{NP*}$ which contradicts $\beta_B^* = 0$ when β^{NP*} is interior.

$$
\begin{array}{rl}
& \lim_{\mu_A \to 0} \alpha \pi_{A,1}(\beta_A, \beta_B, p_0, q_1) + (1-\alpha)\pi_1^{NP}(\beta_A, p_0, q_1) - \pi_0(\beta_A, p_0, q_0) \\
= & 2\omega_0 \left[(2q_1 - 1) \left(\alpha \sum_{e^i \in E} \frac{e^i_{|V=0} e^i_{|V=1}}{Pr(e=e^i)} + 1 - \alpha \right) - (2q_0 - 1) \right]
\end{array} \tag{26}
$$

Using the fact that trader B impacts trader A by rushing with probability β^{NP*} in the limit, if $\beta^{NP*} > 0$, the summation term in (26) contains an informative event. Therefore, given $\alpha > 0$, the term in brackets is strictly increasing for $p_0 > \frac{1}{2}$ as is the case in the proof of Proposition 2. The rest of the proof follows similarly, so that there exists a strictly positive value of μ_A, $\bar{\mu}_{33}$, such that the result holds for $\mu_A \leq \bar{\mu}_{33}$. If $\beta^{NP*} = 0$, $\beta_B^* = 0$ is possible such that trader B has no impact on trader A. But, in this case we must also have $\beta_A^* = \beta^{NP*} = 0$. If, as p_0 increases from one half, β_A^* reaches zero at some $\hat{p}_0 \geq \frac{1}{2}$, then $\pi_1^{NP}(0, \hat{p}_0, q_1) - \pi_0(0, \hat{p}_0, q_0) = 0$. For $p_0 \geq \hat{p}_0$, we have[47]

$$
\begin{array}{rl}
& \pi_1^{NP}(0, p_0, q_1) - \pi_0(0, p_0, q_0) \\
= & \omega_0 \left[(2q_1 - 1) \left(\frac{1}{Pr(a_1 = buy)} + \frac{1}{Pr(a_1 = sell)} \right) - 2(2q_0 - 1) \right]
\end{array} \tag{27}
$$

The sign of (27) is determined by the term in brackets which is easily shown to be increasing in p_0 for $p_0 > \frac{1}{2}$. Therefore, if it is zero at some $\hat{p}_0 \geq \frac{1}{2}$, it is strictly positive for all $p_0 \geq \hat{p}_0$ so that $\beta_A^* = 0$ remains an equilibrium.

Part e). To see that trader A rushes more often as q_0 increases, suppose not: $\beta_A^* \leq \hat{\beta}_A^*$ where $\hat{\beta}_A^*$ is the initial equilibrium probability of rushing prior to the increase. By assumption, $\hat{\beta}_A^* \in (0,1)$ so that trader A waits with positive probability and her trade impacts trader B whenever she does. From the proof of part a), if β_A^* is smaller, β_B^* must be larger. In addition, $\pi_{B,0}$ increases in q_0 (see Lemma A3 of the Online Appendix) and $\pi_{B,1}$ is unaffected, which also causes β_B^* to increase. Therefore, we must have $\beta_B^* \geq \hat{\beta}_B^*$, where $\hat{\beta}_B^*$ is trader B's initial equilibrium probability of rushing prior to the increase. But, the increase in β_B^*, combined with the fact that $\pi_{A,0}$ strictly increases in q_0, then implies $\beta_A^* > \hat{\beta}_A^*$, a contradiction.

Finally, set $\bar{\mu}_3 = \min\{\bar{\mu}_2, \bar{\mu}_{31}, \bar{\mu}_{32}\}$. Then, for $\bar{\mu}_3$ strictly positive, for all $\mu_A, \mu_B \leq \bar{\mu}_3$, the results of parts a) through e) hold.□

Lemma A4: *There exists a strictly positive upper limit on the probability of informed trading, $\bar{\mu} > 0$, such that the results of Propositions 1-4 hold.*

Proof of Lemma A4: Propositions 1, 2, and 4 show that there exist $\bar{\mu}_1 > 0$, $\bar{\mu}_2 > 0$, and $\bar{\mu}_3 > 0$ such that the respective results hold under Assumption 1 (Propositions 1 and 2) or Assumption 2 (Proposition 4). Therefore, at $\bar{\mu} = \min\{\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3\}$, we have $\bar{\mu} > 0$ such that all of the results hold.□

Proof of Proposition 5:

[47]The limiting case of $\mu_A \to 0$ is not helpful here because, in the limit, trader A always has a strict incentive to wait so no such $\hat{p}_0$ exists for any parameterization.

The pricing error with a public belief, $p_2 = E[V|I_2]$ is given by

$$\begin{aligned}
\mathbb{E}_2 &\equiv E\left[(V-p_2)^2\right] \\
&= E\left[E\left[(V-E[V|I_2])^2|I_2\right]\right] \\
&= E\left[E\left[V^2|I_2\right]-E[V|I_2]^2\right] \\
&= E\left[p_2(1-p_2)\right]
\end{aligned}$$

In case of a single trader only, trader A, we have

$$\begin{aligned}
\mathbb{E}_2(\beta_A) &= E\left[p_2(1-p_2)\right] \\
&= E\left[p_2\right]-E\left[p_2^2\right] \\
&= p_0-E\left[p_2^2\right] \\
&= p_0(buy_{0|V1}+buy_{0|V0}+buy_{1|V1}+buy_{1|V0})-E\left[p_2^2\right]
\end{aligned}$$

where the last inequality follows from the martingale property of the public belief ($E[p_2] = p_0$) and the fact that $buy_{0|V1}+buy_{0|V0}+buy_{1|V1}+buy_{1|V0} = 1$. For the expectation of p_2^2, expand over the four possible trades of the trader:

$$\begin{aligned}
& E\left[p_2^2\right] \\
= \; & Pr(a_0 = buy)\left(\tfrac{p_0 buy_{0|V1}}{Pr(a_0=buy)}\right)^2 \\
& + Pr(a_0 = sell)\left(\tfrac{p_0 sell_{0|V1}}{Pr(a_0=sell)}\right)^2 \\
& + Pr(a_1 = buy)\left(\tfrac{p_0 buy_{1|V1}}{Pr(a_1=buy)}\right)^2 \\
& + Pr(a_1^2 = sell)\left(\tfrac{p_0 sell_{1|V1}}{Pr(a_1=sell)}\right)^2
\end{aligned} \tag{28}$$

Consider the sum of the first two terms:

$$\begin{aligned}
& Pr(a_0 = buy)\left(\tfrac{p_0 buy_{0|V1}}{Pr(a_0=buy)}\right)^2 + Pr(a_0 = sell)\left(\tfrac{p_0 sell_{0|V1}}{Pr(a_0=sell)}\right)^2 \\
= \; & \frac{\left(p_0 buy_{0|V1}\right)^2}{p_0 buy_{0|V1}+(1-p_0)buy_{0|V0}} + \frac{\left(p_0 buy_{0|V0}\right)^2}{p_0 buy_{0|V0}+(1-p_0)buy_{0|V1}} \\
= \; & p_0^2\left(\frac{p_0 buy_{0|V0} buy_{0|V1}(buy_{0|V1}+buy_{0|V0})+(1-p_0)\left((buy_{0|V1})^3+(buy_{0|V0})^3\right)}{Pr(a_0=buy)Pr(a_0=sell)}\right) \\
= \; & p_0^2(buy_{0|V1}+buy_{0|V0})\left(\frac{p_0 buy_{0|V0} buy_{0|V1}+(1-p_0)\left((buy_{0|V1})^2-buy_{0|V1}buy_{0|V0}+(buy_{0|V0})^2\right)}{Pr(a_0=buy)Pr(a_0=sell)}\right) \\
= \; & p_0^2(buy_{0|V1}+buy_{0|V0})\left(\frac{buy_{0|V0} buy_{0|V1}+(1-p_0)\left(buy_{0|V1}-buy_{0|V0}\right)^2}{Pr(a_0=buy)Pr(a_0=sell)}\right)
\end{aligned}$$

Using this expression, we have

$$
\begin{aligned}
& p_0(buy_{0|V1} + buy_{0|V0}) - Pr(a_0 = buy)\left(\frac{p_0 buy_{0|V1}}{Pr(a_0=buy)}\right)^2 - Pr(a_0 = sell)\left(\frac{p_0 sell_{0|V1}}{Pr(a_0=sell)}\right)^2 \\
= \;& p_0(buy_{0|V1} + buy_{0|V0})\frac{buy_{0|V1}buy_{0|V0}+p_0(1-p_0)\left(buy_{0|V1}-buy_{0|V0}\right)^2-p_0 buy_{0|V0}buy_{0|V1}-p_0(1-p_0)(buy_{0|V1}-buy_{0|V0})^2}{Pr(a_0=buy)Pr(a_0=sell)} \\
= \;& (buy_{0|V1} + buy_{0|V0})\frac{p_0(1-p_0)buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} \\
= \;& (buy_{0|V1} + buy_{0|V0})\frac{\omega_0 buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)}
\end{aligned}
$$

Doing a similar exercise for the second two terms in (28), we then have

$$
\begin{aligned}
& \mathbb{E}_2(\beta_A) \\
= \;& p_0(buy_{0|V1} + buy_{0|V0} + buy_{1|V1} + buy_{1|V0}) - E\left[p_2^2\right] \\
= \;& (buy_{0|V1} + buy_{0|V0})\frac{\omega_0 buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} \\
& +(buy_{1|V1} + buy_{1|V0})\frac{\omega_0 buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)}
\end{aligned}
$$

The claim is that the reduction in $\mathbb{E}_2(\beta_A)$ due to the informed trader's trades is strictly largest when $p_0 = \frac{1}{2}$. Prior to any trade, the pricing error is ω_0. Therefore, the reduction in the pricing error is given by

$$
\begin{aligned}
\omega_0 - \mathbb{E}_2 \;=\; & \omega_0 \left[1 - (buy_{0|V1} + buy_{0|V0})\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0 = buy)Pr(a_0 = sell)}\right. \\
& \left. -(buy_{1|V1} + buy_{1|V0})\frac{buy_{1|V1}buy_{1|V0}}{Pr(a_1 = buy)Pr(a_1 = sell)}\right]
\end{aligned} \tag{29}
$$

First, note that the reduction in the pricing error is strictly positive because the term in square brackets is easily shown to be strictly positive using the facts that $buy_{0|V1} + buy_{0|V0} + buy_{1|V1} + buy_{1|V0} = 1$, each of $buy_{0|V1} + buy_{0|V0}$ and $buy_{1|V1} + buy_{1|V0}$ is between 0 and 1, and each of the ratio terms is strictly less than one. To show that the reduction in the pricing error reaches a strict maximum at $p_0 = \frac{1}{2}$, consider the partial derivative with respect to p_0 (holding β_A fixed). For the term corresponding to the derivative of ω_0, we have $(1 - 2p_0)$ multiplied by the strictly positive bracketed term of (29). This term is therefore strictly positive for $p_0 < \frac{1}{2}$ and strictly negative for $p_0 > \frac{1}{2}$, reaching a strict maximum at $p_0 = \frac{1}{2}$. For the term corresponding to the derivative of the bracketed term in (29), consider the derivative:

$$
\begin{aligned}
& -\frac{\partial}{\partial p_0}\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)} \\
= \;& -\frac{\partial}{\partial p_0}\frac{buy_{0|V1}buy_{0|V0}}{buy_{0|V1}buy_{0|V0}+\omega_0(buy_{0|V1}-buy_{0|V0})^2} \\
= \;& \frac{buy_{0|V1}buy_{0|V0}(1-2p_0)(buy_{0|V1}-buy_{0|V0})^2}{Pr(a_0=buy)^2 Pr(a_0=sell)^2}
\end{aligned}
$$

which is weakly positive for $p_0 < \frac{1}{2}$ and weakly negative for $p_0 > \frac{1}{2}$ so that $-\frac{buy_{0|V1}buy_{0|V0}}{Pr(a_0=buy)Pr(a_0=sell)}$ reaches a maximum at $p_0 = \frac{1}{2}$ (or is everywhere zero if $\beta_A = 0$). Similarly, $-\frac{buy_{1|V1}buy_{1|V0}}{Pr(a_1=buy)Pr(a_1=sell)}$ reaches a maximum at $p_0 = \frac{1}{2}$ (or is everywhere zero if $\beta_A = 1$). The derivative of the bracketed term then must reach a maximum at $p_0 = \frac{1}{2}$ because both of the derivatives of the ratio terms cannot simultaneously be zero everywhere. Hence, the reduction in the pricing error overall, reaches a strict maximum at $p_0 = \frac{1}{2}$.

For the numerical simulations of Section 4, we require an expression for the equilibrium pricing error in the competitive model of Section 3. From the expression above for the case of a single trader, define

$$\begin{aligned} & W_2(\beta_B, p') \\ = \ & (buy_{B,0|V1} + buy_{B,0|V0}) \frac{p'(1-p')buy_{B,0|V1}buy_{B,0|V0}}{Pr(a_{B,0}=buy)Pr(a_{B,0}=sell)} \\ & +(buy_{B,1|V1} + buy_{B,1|V0}) \frac{p'(1-p')buy_{B,1|V1}buy_{B,1|V0}}{Pr(a_{B,1}=buy)Pr(a_{B,1}=sell)} \end{aligned}$$

as the contribution to the pricing error from trader $B's$ trade at a price, p', where it is understood that the various expressions for the trade probabilities depend upon β_B and p'. We can obtain the overall pricing error by taking the expectation of $W_2(\beta_B, p')$ over the trade (buy or sell in either period) of trader A:

$$\begin{aligned} & \mathbb{E}_2(\beta_A^*, \beta_B^*, \beta^{NP*}) \\ = \ & Pr(a_{A,0} = buy)W_2(\beta^{NP*}, p_{buy_{A,0}}) + Pr(a_{A,0} = sell)W_2(\beta^{NP*}, p_{sell_{A,0}}) \\ & Pr(a_{A,1} = buy)W_2(\beta_B^*, p_{buy_{A,1}}) + Pr(a_{A,1} = sell)W_2(\beta_B^*, p_{sell_{A,1}}) \end{aligned}$$

where $p_{buy_{A,0}} = \frac{p_0 buy_{A,0|V1}}{p_0 buy_{A,0|V1}+(1-p_0)buy_{A,0|V0}}$ is the updated price after a rushed buy by trader A and similarly for $p_{sell_{A,0}}$, $p_{buy_{A,1}}$, and $p_{sell_{A,1}}$. The first two terms correspond to a rushed trade by trader A, in which case trader B rushes with probability, β^{NP*}. The second two terms correspond to a second period trade by trader A in which case trader B rushes with probability, β_B^*. Although for the second two terms trader B may actually trade first, the possible prices, p_2, and therefore the pricing error, do not depend upon the actual order of trades so that we can write the expression as if trader A always trades first. □