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Utility Transversality: A Value-Based Approach

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ABSTRACT

We examine multiattribute decision problems where a value function is specified over the attributes of a decision problem, as is typically done in the deterministic phase of a decision analysis. When uncertainty is present, a utility function is assigned over the value function to represent the decision maker's risk attitude towards value, which we refer to as a value-based approach. A fundamental result of using the value-based approach is a closed form expression that relates the risk aversion functions of the individual attributes to the trade-off functions between them. We call this relation utility transversality. The utility transversality relation asserts that once the value function is specified there is only one dimension of risk attitude in multiattribute decision problems. The construction of multiattribute utility functions using the value-based approach provides the flexibility to model more general functional forms that do not require assumptions of utility independence. For example, we derive a new family of multiattribute utility functions that describes richer preference structures than the usual multilinear family. We also show that many classical results of utility theory, such as risk sharing and the notion of a corporate risk tolerance, can be derived simply from the utility transversality relations by appropriate choice of the value function. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: multiattribute utility; value function; risk attitude; aspiration equivalent

1. INTRODUCTION

In everyday decision situations, we are faced with balancing multiple and sometimes conflicting value attributes. When the decision situation is deterministic, the problem of choosing the best alternative is reduced to that of assigning a value function that specifies the tradeoffs among the attributes. The optimal decision alternative is the one with the largest value as determined by the value function. When uncertainty is present, the value-based approach would take the next step of assigning a utility function over value to represent the decision maker's attitude towards risk (Matheson and Howard, 1968). This approach is also commonly known as the 'Stanford School approach' or the 'value function approach'.

A second approach for constructing multiattribute utility functions, often referred to as the 'Keeney-Raiffa' approach , assesses utility functions directly over the attributes by establishing (or assuming) the applicability of some utility independence conditions to determine (and simplify) the functional form of the multiattribute utility function. For example, when there is mutual utility independence among two attributes, the two-attribute utility function has the well-known multilinear form

$$U(x,y)=k_xU(x)+k_yU(y)+(1-k_x-k_y)U(x)U(y) \tag{1}$$

where k_x and k_y are constants that are derived from the decision maker's assessments (Keeney and Raiffa, 1976). If, in addition, both k_x and k_y are equal to zero, then we have the multiplicative utility form, $U(x,y)=U(x)U(y)$, and if $k_x+k_y=1$, then $U(x,y)=k_xU(x)+k_yU(y)$ and we have the additive utility form. Because of the simplicity of these expressions, the bulk of the literature has focused on assessing individual

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utility functions for each attribute and then combining them in an additive, multiplicative, or multilinear form to obtain a multiattribute utility function. This approach is indeed valid when the appropriate forms of utility independence conditions exist.

In many situations that occur in practice, however, such utility independence conditions may not exist, and we may need to incorporate the interactions between the different attributes in the construction of the multiattribute utility function. Several authors have discussed this issue and have argued for the need to incorporate utility dependence among the attributes of a decision situation when constructing multiattribute utility functions (Kirkwood, 1976, Bell, 1979a,b, Keeney, 1971, 1981; Farquhar and Fishburn, 1982, Harvey, 1993, Abbas and Howard, 2005).

We observe that constructing a multiattribute utility function using a utility function over a value function does not require any assumptions about utility independence. Furthermore, it separates the multiattribute utility assessment into a deterministic multiattribute value assessment and a single assessment of risk attitude over value (or alternately over any other single attribute). As such, it provides a convenient way to model many forms of multiattribute utility functions whether or not utility independence conditions exist.

Our purpose in this paper is to illustrate how to construct general multiattribute utility functions that incorporate utility dependence by using a utility function over the value function. While previous literature discussed the exponential utility function over an additive value function to produce multiplicative utility functions (Keeney and Raiffa, 1976), we wish to more fully develop an approach that also applies to cases where multiplicative or multilinear forms of utility functions are not applicable. We also clarify the risk properties of multiattribute utility functions constructed using utility functions over value functions, and provide a link between the multiattribute utility function constructed using a value function and the utility function assessments over the individual attributes that are consistent with that value function. We show that once the value function is specified there is only one dimension of risk attitude in a multiattribute decision problem. We also derive a closed form expression that relates the risk aversion functions of the individual attributes to the trade-off functions between them. We call this relation *utility transversality*. The utility transversality expression asserts that once the value function is specified, the individual conditional utility functions over the attributes are highly constrained. As a consequence, we need to assess only a single utility function over one of the attributes, or over the value function, and then determine the remaining risk aversion functions (or the remaining conditional utility functions) through the transversality relation.

We end this paper with several applications of the utility transversality relation and show how this approach provides a powerful method to derive many well-known results in the economics literature simply by appropriate choice of the value function.

2. THE CHAIN RULE FOR RISK AVERSION

When uncertainty is present, we assign a utility function over the value function to represent the decision maker's risk attitude

$$U(x_1, \ldots, x_n) = U_V(V(x_1, \ldots, x_n)) \tag{2}$$

where $x_1, \ldots, x_n$ are the attributes of interest, $V(x_1, \ldots, x_n)$ is the deterministic value function, $U_V(V)$ is the utility on value, and the composite function, $U(x_1, \ldots, x_n)$, is the multiattribute utility function.

Our focus in this paper will be on functions $U(x_1, \ldots, x_n)$ and $V(x_1, \ldots, x_n)$ that are twice continuously differentiable with positive first derivatives. Pratt (1964) and Arrow (1965) defined the risk aversion function over a single monetary attribute as

$$\gamma_x^U \triangleq -\frac{U''(x)}{U'(x)} \tag{3}$$

In the case of multiple attributes, the risk aversion function, $\gamma_{x_i}^U$, for a single attribute, $x_i, i = 1, \ldots, n$ is defined in terms of the partial derivatives of the multiattribute utility function

$$\gamma_{x_i}^U \triangleq -\frac{U''_{x_i}}{U'_{x_i}} \tag{4}$$

where $U''_{x_i} = \partial^2 U(x_1, \ldots, x_n)/\partial x_i^2$, and $U'_{x_i} = \partial U(x_1, \ldots, x_n)/\partial x_i$.

We will now focus on the case of two attributes, x and y, but the same results generalize directly to multiple attributes. Consider a utility function over a two-attribute value function,

$$U(x, y) = U_V(V(x, y)) \tag{5}$$

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Taking the first partial derivative of (5) with respect to x and using the chain rule for partial derivatives gives

$$U'_x = \frac{\partial U_V(V(x,y))}{\partial x} = U'_V(V(x,y))V'_x(x,y) = U'_V V'_x \tag{6}$$

where $U'_V \triangleq \mathrm{d}U_V/\mathrm{d}V$ and $V'_x = \partial V(x,y)/\partial x$

Now we use the chain rule again and evaluate the derivative of Equation (6) with respect to x, as

$$\frac{\partial^2 U(x,y)}{\partial x^2} \triangleq U''_x = U''_V \left(V'_x\right)^2 + U'_V V''_x \tag{7}$$

Substituting equations (6) and (7) into (3) gives the risk aversion function for attribute, x, as

$$\gamma_x^U = -\frac{U''_x}{U'_x} = -\frac{U''_V\left(V'_x\right)^2 + U'_V V''_x}{U'_V V'_x} = -\frac{U''_V}{U'_V}V'_x - \frac{V''_x}{V'_x} \tag{8}$$

Let us now define a risk aversion function for the utility function over the value function. We use the notation, γ_V^U, to represent this term, where

$$\gamma_V^U \triangleq -\frac{U''_V}{U'_V} \tag{9}$$

Let us also define a new expression, which has the same form as a risk aversion function but is applied to the value function. For notational convenience, we define this new term as the contribution of the value function to the risk aversion function of attribute x and use the notation γ_x^V, where

$$\gamma_x^V \triangleq -\frac{V''_x}{V'_x} \tag{10}$$

Note that γ_x^V is determined completely by the value function and not by any utility function over the value function or by any risk attitude. However, its expression is similar to that of a risk aversion function and, as we shall see, this term contributes directly to the risk aversion function over each of the individual attributes. The sign of this contribution depends only on the concavity or convexity of the value function with respect to the attribute of interest.

Substituting Equations (9) and (10) into Equation (8) gives *the chain rule for risk aversion*

$$\gamma_x^U = \gamma_V^U V'_x + \gamma_x^V \tag{11}$$

Equation (11) expresses the risk aversion function of an attribute x as a sum of two terms. The first term, $\gamma_V^U V'_x$, is a product of the risk aversion function of utility towards value and the partial derivative of the value function with respect to the attribute of interest, x. The partial derivative, V'_x, is in effect a unit conversion factor. The second term, γ_x^V, is the value function contribution to the risk aversion of the attribute of interest. Dyer and Sarin (1982) used a similar equation for the case of single attributes and defined the difference $\gamma_x^U - \gamma_x^V$ as the relative risk version.

Equation (11) shows that the risk aversion for attribute x is completely specified by knowledge of the risk aversion of utility towards value and other parameters related only to the value function. Since the value function represents deterministic preferences, and can in principle be assessed with deterministic questions, then assessing a utility function over the value function determines the utility function and risk aversion function for all the attributes. Of course, any increasing monotonic function of a value function also represents identical deterministic preferences.

Alternatively, if one of the attributes (and not the value function) is expressed in units that are easy to relate to, such as money, the utility function can be assessed over this single attribute (monetary units also have the additional advantage of permitting important calculations such as buying and selling prices (certain equivalents) and the value of information or control). We can simplify the multiattribute utility assessment by assessing only a single utility function over the attribute that we feel more comfortable reasoning about and describing risk attitude. Once this single utility function is determined, we can then determine the utility function for the remaining attributes from the value function, which will be topic of the next section.

3. UTILITY TRANSVERSALITY

In this section, we build on the previous results and relate the risk aversion functions of the individual attributes to the trade-off functions between them. If we assess a risk aversion function for one attribute, say x, we can re-arrange Equation (11) for the risk attitude on value to obtain

$$\gamma_V^U = \frac{\gamma_x^U - \gamma_x^V}{V'_x} \tag{12}$$

Consistency in the assigned utility values requires that the same risk aversion function be derived

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using any of the attributes. For example, if we equate this expression with a similar one for attribute y, we get

$$\frac{\gamma_x^U - \gamma_x^V}{V_x'} = \frac{\gamma_y^U - \gamma_y^V}{V_y'} \tag{13}$$

Equation (13) relates the risk aversion function of attribute x to the risk aversion function of attribute y. We call this relationship *the utility transversality relation* as it translates risk aversion functions across the different attributes. Furthermore, if we are given the value function and the risk aversion function for attribute y, we can use the transversality relation to solve for the risk aversion function on x as

$$\gamma_x^U = \left[\gamma_y^U - \gamma_y^V\right]\frac{V_x'}{V_y'} + \gamma_x^V \tag{14}$$

Equation (14) may seem difficult to interpret at first, but we can simplify this equation by observing that the value function is constant along an isopreference contour,

$$V(x, y) = \text{Constant along an isopreference contour} \tag{15}$$

As a consequence, the total derivative of the value function across an isopreference contour is zero.

$$\mathrm{d}V(x, y) \triangleq V_x' \,\mathrm{d}x + V_y' \,\mathrm{d}y = 0 \tag{16}$$

Re-arranging (16) gives the tradeoff between the two attributes x and y as

$$\frac{V_x'}{V_y'} = -\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{\text{isopreference contour}} \triangleq t(x, y) \tag{17}$$

where $t(x, y)$ defined above is the deterministic trade-off function between attributes y and x along an isopreference contour. Substituting (17) into (14) yields

$$\gamma_x^U(x, y) = \left[\gamma_y^U(x, y) - \gamma_y^V(x, y)\right] t(x, y) + \gamma_x^V(x, y) \tag{18}$$

To simplify the notation, we will drop the (x, y)terms from equation (18) keeping in mind that the parameters of the equation will generally depend on the values of both x and y. *The utility transversality relation* then becomes

$$\gamma_x^U = (\gamma_y^U - \gamma_y^V)t + \gamma_x^V \tag{19}$$

This equation shows how the attribute utility function on x depends on the attribute utility function on y adjusted only by the mathematical features of the value function. It also shows the potential importance of the risk aversion contributions of the value function. *In the presence of a value function, there is only one independent dimension of risk attitude!* All other risk attitudes are constrained by utility transversality.

Rearranging (19) gives

$$(\gamma_x^U - \gamma_x^V) = (\gamma_y^U - \gamma_y^V)t \tag{20}$$

Equation (20) is another way to relate the risk aversion functions across the different attributes. Note that the ratio of the relative risk aversion functions, the terms in parentheses defined by Dyer and Sarin (1982), is in fact equal to the deterministic trade-off function between the attributes. This is a general result for any multiattribute utility function whether or not mutual utility independence exists. Furthermore, Equation (20) asserts that *assessing the risk aversion functions and the trade-off functions independently may result in inconsistencies in the utility assessment*. We will now work through a full example to illustrate the utility transversality relation between attributes for a decision situation.

3.1. Example: on fates comparable to death

The following example is adapted from (Howard, 1980) on making trade-offs concerning situations where a decision maker is exposed to fates comparable to death (such as outcomes of medical surgery). The decision maker provides a value function over two attributes: consumption, x and health state, y. The health state is a disability level normalized from 0 (instant painless death) to 1 (current health with no disability). The value function over consumption and health states is given as

$$V(x, y) = xy^\eta \tag{21}$$

where x is expressed in dollars, y is the health state, and η determines the trade-off between x and y.

One method to determine the multiattribute utility function is to assess a utility function, $U_V(V)$, over this value function to represent the decision maker's risk attitude towards value. For example, if $U_V(V)$were exponential

$$U_V(V) = U_V(V(x, y)) = 1 - \mathrm{e}^{-\gamma V(x,y)} = 1 - \mathrm{e}^{-\gamma x y^\eta} \tag{22}$$

where $\gamma = \gamma_V^U$ is the (constant) risk aversion coefficient towards value.

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Note that the resulting multiattribute utility function does not exhibit mutual utility independence between the attributes. Assessing a multiattribute utility function that incorporates this level of dependence is a difficult task to perform without first assessing a value function.

Using the chain rule for risk aversion of Equation (11), we can now deduce the risk aversion function towards each of the attributes of the decision problem. For example, in the case of attribute x, we have $V'_x = y^\eta$ and $\gamma_x^V = 0$. Equation (11) reduces to

$$\gamma_x^U = \gamma_V^U V'_x + \gamma_x^V = \gamma y^\eta \tag{23}$$

For attribute y we have $V'_y = \eta x y^{\eta-1}$ and $\gamma_y^V = -(\eta-1)/y$

$$\gamma_y^U = \gamma_V^U V'_y + \gamma_y^V = \gamma \eta x y^{\eta-1} - \frac{\eta-1}{y} \tag{24}$$

Note that while the utility function over value exhibited constant absolute risk aversion, neither of the risk aversion expressions for an attribute in (23) or (24) is a constant. This is a result of the utility dependence between the attributes where the attribute risk aversion is strongly influenced by the contribution of the value function.

There is an important point that needs to be made about transformations on value functions. While a positive monotonic transformation over value provides the same deterministic preference ordering, it does change the multiattribute utility function that is constructed. Consider for example, an exponential utility function over the Cobb–Douglas value function in (22). As we have seen, this utility function leads to utility dependence between the attributes. On the other hand, if the value function is scaled by a logarithmic transformation, we have

$$V_t(x,y) = \log(V(x,y)) = \log(x) + \eta \log(y) \tag{25}$$

If we assign an exponential utility function over this scaled value function, we have

$$\begin{aligned} U(x,y) &= 1 - e^{-\gamma S(x,y)} \\ &= 1 - e^{-\gamma \log(x) - \eta\gamma \log(y)} \\ &= 1 - x^{-\gamma} y^{-\eta\gamma} \end{aligned} \tag{26}$$

The multiattribute utility function over the scaled value function now exhibits mutual utility independence between the attributes, but we have changed the preferences towards risk. In order to maintain the same overall multiattribute utility preferences, we would have to modify the utility function over value to include the inverse transformation.

As a further illustration of the transversality relation, we now show how to use a different approach to construct the multiattribute utility function. Suppose that the decision maker was more comfortable assigning a utility function over the monetary attribute, x, than assigning a utility function over the value function. Let us further assume that his utility function on attribute x is exponential with a constant risk aversion coefficient, γ_o

$$U(x) = 1 - e^{-\gamma_o x} \tag{27}$$

We have $\gamma_x^U = \gamma_o$, $\gamma_x^V(x,y) = 0$, $\gamma_y^V(x,y) = -(\eta-1)/y$, and $t(x,y) = y/\eta x$.

We can now apply the utility transversality relation of (19) to obtain the risk aversion function for attribute y as

$$\gamma_y^U(x,y) = \frac{\gamma_0 \eta x + \eta - 1}{y} \tag{28}$$

For a constant x value in (28), we can now determine the utility function for y using Arrow–Pratt's definition of the risk aversion function,

$$\begin{aligned} \gamma_y^U(x,y) &= -\frac{d}{dy}\ln(U'(x,y)), \text{ where } U'(x,y) \\ &= \frac{d}{dy}U(x,y) \end{aligned} \tag{29}$$

where $U'(x,y) = \frac{\partial}{\partial y}U(x,y)$

Solving (29) gives

$$U(y) = \int_{-\infty}^{y} e^{-\int_{-\infty}^{w} \gamma_y^U(x,z)\, dz}\, dw \tag{30}$$

where w and z are dummy variables.

From (28) and (30), we recognize that the utility function over y must be logarithmic or power as shown below

$$U(y) = \begin{cases} \log(y) & 2 - \eta(\gamma_0 x + 1) = 0 \\ \dfrac{y^{2-\eta(\gamma_0 x+1)}}{2 - \eta(\gamma_0 x + 1)} & \text{otherwise} \end{cases} \tag{31}$$

Assigning any other utility function for y with the given value trade-offs will result in inconsistencies with the axioms of multiattribute utility theory.

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4. CONJUGATE VALUE FUNCTIONS

In the previous example, we have seen that a certain family of utility functions (such as exponential) when assessed over an attribute, x, may induce a different utility function (such as logarithmic) over another attribute, y. Now we introduce the notion of a conjugate value function. A value function, V, is said to be a conjugate to a utility function on value, $U(V)$, if the marginal utility function that is induced on each attribute, $U(x_i)$, is of the same family as $U(V)$. As a result of the transversality relation, the utility functions induced over all of the attributes will be of the same family. As we shall see, this result simplifies the analysis and enables us to determine the form of the utility function over each of the conjugate attributes by inspection. The following example demonstrates that the product value function is conjugate to a logarithmic utility function on value.

4.1. Example: product value function

If we assume the utility function $U(V) = \log(V)$, then from Equation (10) $\gamma_V^U = 1/V$. Now if the value function is a product value function, then

$$V(x, y) = xy,\ V'_x = y,\ V'_y = x, \text{ and } \gamma_x^V = \gamma_y^V = 0 \tag{32}$$

Substituting (32) into the chain rule for risk aversion (11), we get

$$\gamma_x^U = \frac{1}{xy}y + 0 = \frac{1}{x} \tag{33}$$

which means the induced utility function on x is also logarithmic. Similar analysis shows the induced utility function on attribute y is also a logarithmic utility function. Based on this result we say that the product value function is conjugate to the logarithmic utility function.

Keeney and Raiffa (1976, Theorem 6.11), discuss an exponential utility function over an additive value function to produce a multiplicative utility function. We leave it to the reader to show that the additive value function is conjugate to the exponential utility function on value following the proof above.

5. MULTILINEAR VALUE FUNCTIONS

We have defined the term $\gamma_{x_i}^V$ as the contribution of the value function to the risk aversion of an attribute x_i. The question the poses itself now is whether there exist value functions that do not contribute to this attribute risk aversion. If such value functions do exist, then they should satisfy the condition

$$\gamma_{x_i}^V = -\frac{V''_{x_i}(x_1, \ldots, x_i, \ldots, x_n)}{V'_{x_i}(x_1, \ldots, x_i, \ldots, x_n)} = 0, \quad \forall i = 1, \ldots, n \tag{34}$$

If this condition holds for attribute x_i, we define the value function as *risk-contribution neutral with respect to* x_i because it does not contribute to its risk aversion in the transversality equation. If this condition holds for all attributes, we define the value function as *risk-contribution neutral*, because it does not contribute at all to attribute risk aversion.

For two attributes, x and y, if x is *risk-contribution neutral* then,

$$\gamma_x^V = -\frac{V''_x}{V'_x} = 0 \tag{35}$$

The general solution to (35) produces a value function that is risk-contribution neutral with respect to attribute x but not necessarily to attribute y:

$$V(x, y) = xA(y) + B(y) \tag{36}$$

where $A(y)$ and $B(y)$are functions of attribute y only.

If both x and y are risk-contribution neutral then

$$\gamma_x^V = -\frac{V''_x}{V'_x} = 0, \quad \gamma_y^V = -\frac{V''_y}{V'_y} = 0. \tag{37}$$

Integration shows that the value functions meeting this condition are the multilinear family (plus a possible constant, which can be set to zero since it does not change the deterministic preference order of the consequences).

$$V(x, y) = k_x x + k_y y + k_{xy} xy \tag{38}$$

where k_x, k_y, k_{xy} are constants, which can be generalized to n attributes.

Special cases of the multilinear value function are:

(1) Linear additive value functions: $V(x, y) = ax + by$, where a and b are given constants.
(2) Multiplicative value functions: $V(x, y) = xy$.

In all these cases, the value function is a linear function in each attribute and does not contribute

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to its induced risk aversion. Multilinear value functions, therefore, only scale the value curve of an attribute when the other attributes change from one point to the other. Extending the multilinear family by adding a constant to any of the above value functions preserves deterministic preferences (comparisons between alternatives) while shifting the base value. This extended multilinear family is the complete set of value functions that are risk-contribution neutral since they are the general solution to the partial differential equation of (37).

Multilinear value functions have been discussed in other contexts in the literature. For example, Dyer and Sarin (1979) showed preferential independence conditions under which value functions take the multilinear form. von Winterfeldt and Edwards (1986) also discussed multilinear value functions as models for aggregating single-attribute value functions.

For multilinear value functions, the chain rule formula for risk aversion reduces to:

$$\gamma^U_{x_i} = \gamma^U_V V'_{x_i} \tag{39}$$

and the transversality relation of Equation (19) reduces to the simple expression

$$\gamma^U_x = \gamma^U_y t \tag{40}$$

In the multilinear case, the ratio between any two of the induced single-attribute risk aversion functions is always the slope of the trade-off function between the two attributes, which can be thought of as a simple unit conversion at the operating point. Again, this result shows how the specification of one dimension of risk attitude determines the risk attitude of the others through the value function trade-offs. Equation (40) can also be reversed as another method to assess the trade-off function between attributes having multilinear value functions: the ratio of the two risk aversion functions determines the trade-off slope between them.

6. APPLICATIONS OF UTILITY TRANSVERSALITY

Having shown that there is only one dimension of risk attitude in multiattribute problems and the relation between the risk aversion functions of different attributes and their trade-off function, we now present several applications of the utility transversality relationships, and show how they easily unify several classical results in the literature.

6.1. Example: time preference

Now let us consider a decision situation with two attributes: x = money received today, and y = money received a year from today. The value function for this situation is chosen to be the net present value and is given as

$$V(x, y) = x + \beta y \tag{41}$$

The term β is the time preference or personal discount rate for 1 year.

Note, that this value function is risk-contribution neutral with $\gamma^V_x(x, y) = 0$ and $\gamma^V_y(x, y) = 0$. Using (40), we have

$$\frac{\gamma^U_x(x, y)}{\gamma^U_y(x, y)} = -\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\beta} \tag{42}$$

Re-arranging gives

$$\gamma^U_y = \beta \gamma^U_x \tag{43}$$

Equation (43) provides us with a relation between the risk aversion function today and the risk aversion function in 1 year. It shows that the risk tolerance should be compounded at the time preference rate. Note that this relationship does not depend on the form of utility function that is assigned to the value function, and therefore generalizes the classic results in the literature that deal only with the NPV and exponential utility functions. We have derived a more general result in a simple manner using the transversality relation. For the NPV function, if the risk aversion coefficient over one attribute is constant (exponential utility function), then the risk aversion coefficient over any other attribute is also a constant (exponential utility function) since the additive value function is conjugate to the exponential utility function.

6.2. Example: risk preference of a partnership

In this example we will demonstrate how the transversality results can also be applied to relate the risk aversion function of a partnership in terms of the risk aversion function of an individual. Let us assume the share of individual i is α_i times the share of the partnership, i.e.

$$V_i = \alpha_i V_P \tag{44}$$

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where V_i is the profit of individual i from the deal, α_i = individual i's percentage of the deal, and V_p is the profit of the whole partnership.

The value function for individual $i$is a linear scaling of the value of the partnership. This is a risk-contribution neutral value function. From (40), we have

$$\gamma_P = \alpha_i \gamma_i \tag{45}$$

Taking the reciprocal of Equation (45), we get

$$\rho_i = \alpha_i \rho_P \tag{46}$$

This result indicates that the fraction of the deal taken by an individual in a partnership should be equal to the ratio of his risk tolerance to the sum of risk tolerances of the partners. This is a classic result in risk sharing literature (Wilson, 1968) and has been derived here using the utility transversality relations. Note also that the transversality relations assert that if the individual utility functions are exponential (have constant risk tolerance), then the group utility function must also be exponential. Once again, we can arrive at this result by observing that the exponential utility function is conjugate to the additive value function.

6.3. Example: relating corporate to divisional risk tolerance

Let us now apply our results to the value function of a company, V_C, derived from the value functions of different divisions within a company. The value function of the company is the sum of the values of different divisions

$$V_C = \sum_{i=1}^{n} V_i \tag{47}$$

Once again, the value transversality relations apply and in this special case we have

$$\gamma_i^U = \gamma_j^U = \gamma_{V_C}^U, \quad \forall i,j \tag{48}$$

This result implies that all divisions should act with the same risk tolerance. The corporate risk tolerance should be used for all corporate decisions.

However, if one of the divisions, k, is owned in partnership with another company, with ownership fraction, α, then using Equation (45) we get

$$\gamma_k^U = \alpha \gamma_{V_C}^U \tag{49}$$

This implies that division, k, should have a lower risk aversion coefficient than the other divisions by a factor of α.

7. SCALED ATTRIBUTES

Sometimes it is natural (or easier) to scale attributes by a certain transformation before combining them in a value function. The scaling transformation may be a linear or a non-linear function of the attribute. For example, one might measure or assess the speed of an object but place value on the kinetic energy, which is proportional to speed squared. In a consumption example, one might assess the amount of food available but have diminishing increase in satisfaction as the quantity increases. Cases like these can be treated with scaling (transformation) functions, where we replace the attribute, x_i, with the scaled attribute, $S_i(x_i)$. Thus the value function becomes $V(S_1(x_1), S_2(x_2), \ldots, S_N(x_N))$.

Taking derivatives as we did in deriving the chain rule, we arrive at

$$\gamma_{x_i}^V = \gamma_{s_i}^V V'_{S_i} + \gamma_{x_i}^{S_i} \tag{50}$$

Equation (50) can be combined with the chain rule for risk aversion of Equation (11) or the transversality relation of Equation (19) to generalize these equations to the case of scaled attributes. The chain rule for risk aversion now becomes

$$\gamma_{x_i}^U = \gamma_V^U V'_{S_i} S'_{x_i} + \gamma_{S_i}^V S'_{x_i} + \gamma_{x_i}^{S_i} \tag{51}$$

where, $\gamma_{x_i}^{S_i} = -S''_{x_i}/S'_{x_i}$.

As before, we note that this is the sum of the contributions to the risk aversion functions of each of the compounded functions, scaled into common units by the appropriate partial derivatives.

For the case of scaled attributes the transversality relation now becomes

$$\frac{\gamma_{x_i}^U - \gamma_{S_i}^V S'_{x_i} - \gamma_{x_i}^{S_i}}{V'_{S_i} S'_{x_i}} = \frac{\gamma_{x_j}^U - \gamma_{S_j}^V S'_{x_j} - \gamma_{x_j}^{S_j}}{V'_{S_j} S'_{x_j}} \tag{52}$$

Solving for the risk aversion function for attribute i yields

$$\gamma_{x_i}^U = \left(\gamma_{x_j}^U - \gamma_{S_j}^V S'_{x_j} - \gamma_{x_j}^{S_j}\right) t - \gamma_{S_i}^V S'_{x_i} - \gamma_{x_i}^{S_i}, \tag{53}$$

where $t(x_i, x_j) = (V'_{S_{x_i}}/V'_{S_{xj}})(S'_{x_i}/S'_{x_j}/) = (\mathrm{d}x_j/\mathrm{d}x_i)$, the tradeoff function between the two attributes as defined before.

Returning to the scaled attributes of Equation (51), if the value function is risk-contribution neutral, then $\gamma_{S_i}^V = 0$; the chain rule becomes

$$\gamma_{x_i}^U = \gamma_V^U V'_{S_i} S'_{x_i} + \gamma_{x_i}^{S_i} \tag{54}$$

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and the transversality relation becomes

$$\gamma^U_{x_i} = \left(\gamma^U_{x_j} - \gamma^{S_j}_{x_j}\right)t - \gamma^{S_i}_{x_i} \quad (55)$$

Furthermore, for the case of risk-contribution neutral value functions, $\gamma^V_{s_i} = 0$, and (50) becomes

$$\gamma^V_{x_i} = \gamma^{S_i}_{x_i} \quad (56)$$

In this case the contribution of the value function towards the risk aversion of any attribute is determined only by the scaling function for that attribute.

In addition, if the utility on value, $U_V(V)$, were risk neutral, the scaling functions would fully determine the risk attitude of each attribute. In this case, the two-attribute utility function takes the form

$$U_V(V(x,y)) = V(x,y) = k_x S_x(x) + k_y S_y(y) + k_{xy} S_x(x) S_y(y) \quad (57)$$

Here, the scaling transformation functions serve as individual attribute utility functions $S_i(x_i) = U_i(x_i)$ that determine the risk attitude towards each attribute. Thus a risk-neutral utility function, $U_V(V)$, on a risk-contribution neutral value function generates the whole family of multilinear utility functions by scaling the attributes with their individual utility functions (Keeney and Raiffa, 1976). For example, consider a small company that generates power with wind turbines and sells its energy to the local electric utility at an uncertain price, P. The amount of energy it produces is a non-linear function of wind speed (which we will assume squared for simplicity). Even if the company is risk neutral over value, it is risk seeking over wind speed because of the risk-contribution of the scaling function.

It is natural to extend this family by incorporating a general utility function on value, $U_V(V)$. We call this new family the *composite multilinear family* of multiattribute utility functions. This family comprises individual scaling functions on each attribute combined with a multilinear value function composed with a general utility function on value,

$$U(x_1, \ldots, x_n) = U_V(V_m(S_1(x_1), \ldots, S_n(x_n))) \quad (58)$$

where $V_m(\ldots)$is a multilinear (risk-contribution neutral) value function.

By adding the final utility function on value, the most commonly used and well-studied multilinear, multiattribute utility function is expanded to allow utility dependencies through a hierarchically structured set of functions.

7.1. Example: revisiting on fates comparable to death

We return to an example we addressed earlier. We can now view the two attributes x and y as scaled by the following scaling functions

$$S_x(x) = x, \quad S_y(y) = y^\eta \quad (59)$$

with the resulting value function

$$V_m(S_x(x), S_y(y)) = S_x(x) S_y(y) \quad (60)$$

which is a risk contribution neutral value function on the scaling transformation functions $S_x(x)$ and $S_y(y)$.

For a risk neutral decision maker over value,

$$U_V(V_m(x,y)) = V_m(x,y) = S_x(x) S_y(y) \quad (61)$$

It is clear that the scaling functions determine the risk attitude of each attribute, and the attributes exhibit mutual utility independence. Using the chain rule of (54), we have

$$\gamma^U_x = \gamma^V_x = \gamma^V_{s_x} V'_{S_x} + \gamma^{S_x}_x = \gamma^{S_x}_x = 0 \quad (62)$$

$$\gamma^U_y = \gamma^V_y = \gamma^V_{s_y} V'_{S_y} + \gamma^{S_y}_y = \gamma^{S_y}_y = \frac{-(\eta - 1)}{y} \quad (63)$$

For an exponential utility function over value, we have

$$U(x,y) = U_V(V_m(S_x(x), S_y(y))) = 1 - e^{-\gamma S_x(x) S_y(y)} \quad (64)$$

which exhibits utility dependence for both the scaled and the original attributes.

From (54), the risk aversion functions are

$$\gamma^U_x = \gamma^U_V V'_{S_x} S'_x + \gamma^{S_x}_x = (\gamma_0)(y^\eta)(1) + 0 = \gamma_0 y^\eta \quad (65)$$

$$\gamma^U_y = \gamma^U_V V'_{S_y} S'_y + \gamma^{S_y}_y = (\gamma_0)(x)(\eta y^{\eta-1}) - \frac{(\eta - 1)}{y} \quad (66)$$

which match our previous results.

This example demonstrates an application of the composite multilinear family of multiattribute utility functions by simplifying the original value function into a multilinear value function with scaled attributes. Furthermore, the example shows a very tractable way of incorporating utility dependence.

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8. CONCLUSIONS

We can now explain the title: the transversality relation strongly links together risk attitudes across attributes. In mathematics, a transverse is a line that intersects a system of other lines. In the presence of a value function there is only one independent dimension of risk attitude, and the transversality relation couples the system of individual risk attitudes. All of the results in this paper are developed using the transversality expression and the related chain rule for risk attitude.

We discussed constructing multiattribute utility functions using utility functions over value functions. We showed how this approach can be applied to several forms of value functions and utility functions over them to yield multiattribute utility functions that incorporate utility dependence. While most applications of multiattribute utility have focused on additive, multiplicative, and multilinear utility forms because of their ease of expression, the utility function over value provides a convenient method to incorporate utility dependence among the attributes using a utility assessment over a single attribute and the transversality relations.

We demonstrated that once the value function is specified there is only one dimension of risk attitude. Hence, we provided a link between the multiattribute utility function constructed using a value function and the utility function assessments over the individual attributes that are consistent with that value function. We illustrated the use of the transversality relations in the unification of several classical results in the literature. Finally, from the multilinear value family we defined a new family of composite multiattribute utility functions that incorporate utility dependence.

REFERENCES

Abbas AE, Howard RA. 2005. Attribute dominance utility. *Decision Analysis* **2**(4): 185–206.

Arrow KJ. 1965. The theory of risk aversion. In *Aspects of the Theory of Risk-Bearing*, Lecture 2. Yrjo Jahnssonin Saatio: Helsinki.

Bell DE. 1979a. Multiattribute utility functions: decompositions using interpolation. *Management Science* **25**: 744–753.

Bell DE. 1979b. Consistent assessment procedures using conditional utility functions. *Operations Research* **27**: 1054–1066.

Dyer JS, Sarin RK. 1979. Measurable multiattribute value functions. *Operations Research* **27**(4): 810–822.

Dyer JS, Sarin R. 1982. Relative Risk Aversion. *Management Science* **28**: 875–886.

Farquhar P, Fishburn P. 1982. Finite degree utility independence. *Mathematical Operation Research* **7**: 348–353.

Harvey CM. 1993. Multiattribute risk linearity. *Management Science* **39**: 389–394.

Howard RA. 1980. On making life and death decisions. In *The Principles and Applications of Decision Analysis*, vol. II, Howard RA, Matheson JE (eds). Strategic Decisions Group: Menlo Park, CA, USA.

Keeney RL. 1971. Utility independence and preferences for multiattributed consequences. *Operations Research* **19**: 875–893.

Keeney RL, Raiffa H. 1976. *Decisions with Multiple Objectives*. Wiley: New York.

Keeney R. 1981. Analysis of preference dependencies among objectives. *Operations Research* **29**: q1105–1120.

Kirkwood CW. 1976. Parametrically dependent preferences for multiattributed consequences. *Operations Research* **7**: 92–103.

Matheson JE, Howard RA. 1968. An introduction to decision analysis. In *The Principles and Applications of Decision Analysis*, vol. I, Howard RA, Matheson JE (eds). Strategic Decisions Group: Menlo Park, CA, USA.

Pratt J. 1964. Risk aversion in the small and in the large. *Econometrica* **32**: 122–136.

von Winterfeldt DW, Edwards. 1986. *Decision Analyses and Behavioral Research*. Cambridge University Press: Cambridge, UK.

Wilson R. 1968. The theory of syndicates. *Econometrica* **36**: 119–132.

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