Interest Rate Volatility and No-Arbitrage Affine Term Structure Models*

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Abstract

An important aspect of any dynamic model of volatility is the requirement that volatility be positive. We show that for no-arbitrage affine term structure models, this admissibility constraint gives rise to a tension in simultaneous fitting of the physical and risk-neutral yields forecasts. In resolving this tension, the risk-neutral dynamics is typically given more priority, thanks to its superior identification. Consequently, the time-series dynamics are derived partly from the cross-sectional information; thus, time-series yields forecasts are strongly influenced by the no-arbitrage constraints. We find that this feature in turn underlies the well-known failure of these models with stochastic volatility to explain the deviations from the Expectations Hypothesis observed in the data.

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1 Introduction

One of the key challenges for stochastic volatility models of the term structures, as observed by Dai and Singleton (2002), is the "tension in matching simultaneously the historical properties of the conditional means and variances of yields." Similarly, Duffee (2002) notes that the overall goodness of fit "is increased by giving up flexibility in forecasting to acquire flexibility in fitting conditional variances." Although the difficulty in matching both first and second moments in affine term structure models has been a robust finding in the literature, the exact mechanism that underlies this tension is not well understood. In this paper, we show that the key element in understanding the tension between first and second moments is the no-arbitrage restriction inducing the additional requirement to match first moments under the risk-neutral distribution. Moreover, we show that precise inference about the risk-neutral distribution has a number of important implications for stochastic volatility term structure models.

The literature has largely attributed the failures of stochastic volatility term structure models to match key properties in the data as the tension between the physical first and second moments. To see the importance of the no-arbitrage constraints, consider, for example, the deviations from the expectations hypothesis (EH). Campbell and Shiller (1991) show that when the EH holds, a regression coefficient of $\phi_n = 1$ should be obtained in the regression

$$y_{n-1,t+1} - y_{n,t} = \alpha_n + \phi_n \left(\frac{y_{n,t} - y_{1,t}}{n-1} \right) + \epsilon_{n,t+1}, \tag{1}$$

where $y_{n,t}$ is the n-month yield at time t. However, in the data, the empirical ϕ_n coefficient estimates are all negative and increasingly so with maturity. Dai and Singleton (2002) (hereafter DS) show that no-arbitrage models with constant volatility are consistent with the downward sloping pattern in the data. However, the no-arbitrage models with one or two stochastic volatility factors are unable to match the pattern in the data. Their results are replicated in Figure 1.[1] DS conjecture "the likelihood function seems to give substantial weight to *fitting volatility* at the expense of matching [deviations from the EH]".

We estimate stochastic volatility factor models that do not impose no arbitrage but fit stochastic volatility of yields. In stark contrast to the no arbitrage models, the stochastic volatility factor models can almost perfectly match the empirical patterns of bond risk premia as characterized by regression coefficients. This finding clarifies that fitting stochastic volatility is not an issue *per se*. Rather, it is the restrictiveness associated with the no-arbitrage structure that underlies the well documented failure of the no arbitrage stochastic volatility models to rationalize the deviations from the EH in the data.

The tension between first and second moments arises because of the fact that volatility must be a positive process. This requires that forecasts of volatility must also be positive. This introduces a tension between first and second moments. This type of tension, observed by Dai and Singleton (2002) and Duffee (2002), is generally present in affine stochastic volatility models, even when no arbitrage restrictions are not imposed. In a no arbitrage

[1] See Section 5 for additional details on the data and our estimation.

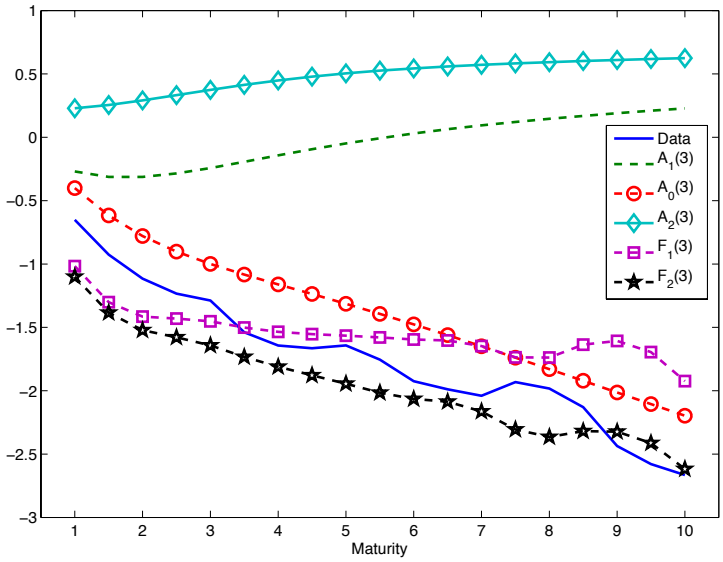


Figure 1: Violations of the Expectations Hypothesis. This figures plots the coefficients ϕ_n from the Campbell-Shiller regression in (1). When risk premia are constant so that the expectations hypothesis holds, the coefficients should be uniformly equal to one across all maturities. The models $A_m(3)$ are three factor no arbitrage models with $m = 0, 1$, or 2 factors driving volatility. The models $F_m(3)$ are three factor models that do not impose no arbitrage with $m = 1$ or 2 factors driving volatility.

model, volatility must also be a positive process under the risk-neutral measure. This induces an additional tension with risk-neutral first moments. This creates a three-way tension now between first moments under the physical and risk-neutral measure and second moments. The relative importance of these moments (and their role in the tension) are determined by the precision with which they can be estimated.

At the heart of our result is the fact that the $\mathbb{Q}$ dynamics is estimated much more precisely than its historical counterpart. Intuitively, although we have only one historical time series with which to estimate physical forecasts, each observation of the yield curve directly represents a term structure of risk neutral expectations of yields. Due to this asymmetry, it is typically "costly" for standard objective functions to "give up" cross-sectional fits for time-series fits in estimation. As a result, when faced with the "first moments" tension – the trade-off between fitting time series and risk-neutral forecasts – standard objective functions typically settle on a rather uneven resolution in which cross-sectional pricing errors are highly optimized at the expense of fits to time series forecasts. The resulting impact on the time series dynamics in turn deprives the estimated model of its ability to replicate the CS regressions – meant to capture the times series properties of the data.

Our findings add to the recent discussion that suggests that no arbitrage restrictions are completely or nearly irrelevant for the estimation of Gaussian dynamic term structure models

(DTSM).[2] Still left open by the existing literature is the question of whether the no arbitrage restrictions are useful in the estimation of DTSMs with stochastic volatility. Our results show that the answer to this question is a resounding yes – an answer that is surprising (given the existing evidence regarding Gaussian DTSMs) but can now be intuitively explained in light of our results. That is, the "first moments" tension essentially provides a channel through which relatively more precise $\mathbb{Q}$ information will spill over and influence the estimation of the $\mathbb{P}$ dynamics. This channel does not exist in the context of Gaussian DTSMs in which the admissibility constraint ensuring positive volatility is not needed.

Our findings also help clarify the nature of the relationship between the no arbitrage structure and volatility instruments extracted from the cross-section of bond yields documented by several recent studies.[3] For example, we show that for the $A_1(N)$ class of models (an N factor model with a single factor driving volatility), the cross-section of bonds will reveal up to N linear combinations of yields, given by the N left eigenvectors of the risk neutral feedback matrix ($K_1^{\mathbb{Q}}$), that can serve as instruments for volatility. The no arbitrage structure then essentially implies nothing more for the properties of volatility beyond the assumed one factor structure and the admissibility conditions. Furthermore, we show that the estimates of $K_1^{\mathbb{Q}}$ are very strongly identified and essentially invariant to volatility considerations. For a variety of sampling and modeling choices, we show that the estimates of $K_1^{\mathbb{Q}}$ are virtually identical across models with or without stochastic volatility.[4] This invariance implies the striking conclusion that a Gaussian term structure model – with constant volatility – can reveal which instruments would be admissible for a stochastic volatility model.[5] An elaborate example illustrating this point is provided in Section 5.2.

Finally, our results help identify aspects of model specifications that may or may not have any significant bearing on the model implied volatility outputs. For example, we show that within the $A_1(N)$ class of models, different specifications of the market prices of risks are unlikely to significantly affect the identification of the volatility factor. To see this, recall from the preceding paragraph that volatility instruments for an $A_1(N)$ model are determined by left eigenvectors of the risk neutral feedback matrix. Intuitively, since the market prices of risks serve as the linkage between the $\mathbb{P}$ and $\mathbb{Q}$ measures, and since the $\mathbb{Q}$ dynamics is very strongly identified, different forms of the market prices of risks are most likely to result in different estimates for the $\mathbb{P}$ dynamics while leaving estimates of risk neutral feedback matrix essentially intact. This thus implies that volatility instruments are likely identical across these models with different risk price specifications. Our intuition is consistent with the

[2] See, for example, Duffee (2011), Joslin, Singleton, and Zhu (2011), and Joslin, Le, and Singleton (2012).

[3] For example, Collin-Dufresne, Goldstein, and Jones (2009) find an extracted volatility factor from the cross-section of yields through a no arbitrage model to be negatively correlated with model-free estimates. Jacobs and Karoui (2009) in contrast generally find volatility extracted from affine models are generally positively related though in some cases they also find a negative correlation. Almeida, Graveline, and Joslin (2011) also find a positive relationship.

[4] In addition to our results, findings by Campbell (1986) and Joslin (2013b) also suggest that risk neutral forecasts of yields are largely invariant to any volatility considerations.

[5] A practical convenience of this result is that we can use the Gaussian model to generate very good starting points for the $A_m(N)$ models. In our estimation, these starting values take only a few minutes to converge to their global estimates.

almost identical performances of volatility estimates implied by $A_1(3)$ models with different (completely affine and essentially affine) risk price specifications as reported in Jacobs and Karoui (2009).

The rest of the paper is organized as follows. In Section 2, we provide the basic intuition as to how the "first moments" tension arises. In Section 3, we lay out the general setup of the term structure models with stochastic volatility that we subsequently consider. Section 4 empirically evaluates the admissibility restrictions under both the physical and risk neutral measures. Section 5 provides a comparison between the stochastic volatility and pure gaussian term structure models. Section 6 examines the impact of no arbitrage restrictions on various' model performance statistics. Section 8 provides some extensions. Section 9 concludes.

2 Basic Intuition

In this section, we develop some basic intuition for our results before elaborating in more detail both theoretically and empirically. We first describe three basic moments that a term structure model should match. We then show how tensions arise in a no arbitrage term structure model in matching those moments. In particular, we show that the presence of stochastic volatility induces a tension between matching first moments under the historical distribution ($\mathbb{P}$) and the risk-neutral distribution ($\mathbb{Q}$). This tension accentuates the difficulty in matching first and second moments under the historical distribution.

2.1 Moments in a term structure model

A term structure model should match:

1. $M_1(\mathbb{P})$**:** the conditional first moments of yields under the historical distribution,

2. $M_1(\mathbb{Q})$**:** the conditional first moments of yields under the risk-neutral distribution, and

3. M_2**:** the conditional second moments of yields.[6]

A number of basic stylized facts are well-known about these moments (see, Piazzesi (2010) or Dai and Singleton (2003), for example.) Empirically, the slope and curvature of the yield curve (as well as the level to a slight extent) exhibit some amount of mean reversion. Also, an upward sloping yield curve often predicts (slightly) lower interest rates in the future. $M_1(\mathbb{P})$ should capture these types of patterns. Recall that risk-neutral forecasts are convexity-adjusted forward rates and therefore matching first moments under the risk-neutral measure, $M_1(\mathbb{Q})$, is closely related to the ability of the model to price bonds. The volatility

[6] We make no distinction between second moments under the historical and risk-neutral distribution though this is possible in some contexts. In Section 8.2 we discuss also the case where there is unspanned stochastic volatility.

of yields is time-varying and persistent. Volatility is also related at least partially to the level and shape of yield curve.[7] M_2 should deliver such features of volatility.

It is worth comparing that we could equivalently replace $M_1(\mathbb{Q})$ with matching risk premia. Dai and Singleton (2003) and others take this approach. In this context, the model should match time-variation in expected excess returns found in the data such as the fact that when yield curve is upward sloping, excess returns for holding long maturity bonds are on average higher. Since excess returns are related to differences between actual and risk-neutral forecasts (i.e. the expected excess return is the difference between an expected future spot rate and a forward rate), such an approach is equivalent to our approach. As we explain later, focusing on risk-neutral expectations has the benefit of isolating parameters which are both estimated precisely and, importantly, invariant to the volatility specification.

2.2 The first moments tension

We now develop some intuition for how the "first moments" tension—that is a tension between matching $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$—arises.

Consider the affine class of models, $A_M(N)$, formalized by Dai and Singleton (2000). Due to the affine structure, the processes for the first N principle components of the yield curve (e.g., level, slope, and curvature), denoted $\mathcal{P}$, can be written as:

$$d\mathcal{P}_t = (K_0 + K_1\mathcal{P}_t)dt + \sqrt{\Sigma_t}dB_t, \tag{2}$$

$$d\mathcal{P}_t = (K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}}\mathcal{P}_t)dt + \sqrt{\Sigma_t}dB_t^{\mathbb{Q}}, \tag{3}$$

where B_t, $B_t^{\mathbb{Q}}$ are standard Brownian motions under the historical measure, $\mathbb{P}$, and the risk neutral measure, $\mathbb{Q}$, respectively. Σ_t is the diffusion process of $\mathcal{P}_t$, taking values as an $N \times N$ positive semi-definite matrix:[8]

$$\Sigma_t = \Sigma_0 + \Sigma_1 V_{1,t} + \ldots + \Sigma_M V_{M,t}, \quad \text{and} \quad V_{i,t} = \alpha_i + \beta_i \cdot \mathcal{P}_t, \tag{4}$$

where $V_{i,t}$'s are strictly positive volatility factors and conditions are imposed to maintain positive semi-definite (psd) Σ_t.[9]

[7] A rich body of literature has shown that the volatility of the yield curve is, at least partially, related to the shape of the yield curve. For example, volatility of interest rates is usually high when interest rates are high and when the yield curve exhibits higher curvature (see Cox, Ingersoll, and Ross (1985), Litterman, Scheinkman, and Weiss (1991), and Longstaff and Schwartz (1992), among others).

[8] Importantly, diffusion invariance implies that *the diffusion, Σ_t, is the same under both measures.* Since Σ_t is the same under both the historical and risk-neutral measures, it must be that the coefficients in (4) are the same under both measures. A caveat applies that at a finite horizon, there may be difference in the coefficients in (4). These arise because of differences in $E_t[V_{t+\Delta t}]$ and $E_t^{\mathbb{Q}}[V_{t+\Delta t}]$. Importantly, however, $(\alpha_i^{\Delta t}, \beta_i^{\Delta t})$ will *not* depend on $\mathbb{P}$ or $\mathbb{Q}$. This differences will manifest in differences in the other coefficients. That is, there will be $(\Sigma_0^{\Delta t,\mathbb{Q}}, \Sigma_1^{\Delta t,\mathbb{Q}}, \ldots, \Sigma_M^{\Delta t,\mathbb{Q}})$ which will be different from $(\Sigma_0^{\Delta t,\mathbb{P}}, \Sigma_1^{\Delta t,\mathbb{P}}, \ldots, \Sigma_M^{\Delta t,\mathbb{P}})$. These differences will not be important for our analysis. Even so, in typical applications, the time horizon is small (from daily to at most one quarter), so even these differences will be minor. See also Section 4 and Appendix B.

[9] Alternatively, one could express the diffusion as $\Sigma_t = \tilde{\Sigma}_0 + \tilde{\Sigma}_1\mathcal{P}_{1,t} + \ldots + \tilde{\Sigma}_N\mathcal{P}_{N,t}$. When the model falls in the $A_M(N)$ class, the matrices $(\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_N)$ will lie in an M-dimensional subspaces, allowing the representation in (4)

The one-factor structure of volatility

For the sake of clarity, let us first specialize to the case: $M = 1$. Due to the positivity of the one volatility factor, $V_t = \alpha + \beta \cdot \mathcal{P}_t$ (where for simplicity we drop the indices in equation (4)), forecasts of V_t at all horizons must remain positive. Thus, to avoid negative forecasts, the $(N-1)$ non-volatility factors must not be allowed to forecast V_t. This in turn requires that the drift of V_t must depend on only V_t.

According to equation (2), the drift of V_t (ignoring constant) is given by $\beta' K_1 \mathcal{P}_t$. For this to depend only on V_t, and thus $\beta' \mathcal{P}_t$, it must be the case that $\beta' K_1$ is a multiple of β'. That is, β must be a left-eigenvector of K_1. Equivalently, β must be an eigenvector of K_1'.

Likewise, applying similar logic under the risk-neutral measure, it must follow that β is a left-eigenvector of $K_1^{\mathbb{Q}}$. Thus, the volatility loading vector β must be a left eigenvector to both the risk neutral feedback matrix, $K_1^{\mathbb{Q}}$, and physical feedback matrix, $K_1^{\mathbb{P}}$. This establishes a tight connection between the physical and risk neutral yields forecasts since $K_1^{\mathbb{P}}$ and $K_1^{\mathbb{Q}}$ are forced to share one common left eigenvector.

With this in mind, an unconstrained estimate of $K_1^{\mathbb{P}}$, for example one obtained by fitting $\mathcal{P}$ to a VAR(1) analogous to (2), may not be optimal. The reason being, such an unconstrained estimate might force $K_1^{\mathbb{Q}}$ to admit a left eigenvector of K_1 as one of its own. Such an imposition can result in poor cross-sectional fits. Likewise, an unconstrained estimate of $K_1^{\mathbb{Q}}$ can significantly impact the time series dynamics, by imposing one of its own left eigenvectors upon $K_1^{\mathbb{P}}$. By stapling the $\mathbb{P}$ and $\mathbb{Q}$ forecasts together, the common left eigenvector constraint potentially triggers some tradeoff as the $\mathbb{P}$ and $\mathbb{Q}$ dynamics "compete" to match $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$.

More general settings

More generally, since the volatility factors $V_{i,t}$ must remain positive, their conditional expectations at all horizons must be positive. For given β_i's, only some values of (K_0, K_1) will induce positive forecasts of $V_{i,t}$ for all possible values of $\mathcal{P}_t$.[10] This is the well-documented tension between matching first and second moments ($M_1(\mathbb{P})$ and M_2) seen in the literature. We would like to choose a particular volatility instrument (β_i's) to satisfy M_2, but the best choice of β_i's to match M_2 may rule out the best choice of (K_0, K_1) to match $M_1(\mathbb{P})$.

Even within an affine factor model with stochastic volatility (that is, a factor model that does not impose conditions for no arbitrage so that (2) applies but not (3)), this tension would arise. That is, no arbitrage does not directly affect this tension. However, for no-arbitrage affine term structure models, the above logic applies equally to both the $\mathbb{P}$ and $\mathbb{Q}$ measures. As before, for a given choice of β_i's, we will be restricted on the choice of $(K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$, so that the drift of $V_{i,t}$ under the risk-neutral measure guarantees that risk-neutral forecasts of $V_{i,t}$ remain positive. Thus the no arbitrage structure adds a tension between M_2 and $M_1(\mathbb{Q})$. That is, the best choice of β_i's to match M_2 may be incompatible with the best choice of $(K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$ to match $M_1(\mathbb{Q})$.

[10] In the affine model we consider, the possible values of $\mathcal{P}_t$ will be an affine transformation of $\mathbb{R}_+^M \times \mathbb{R}^{N-M}$ for some (M, N).

This implies a three-way tension between $M_1(\mathbb{P})$, $M_1(\mathbb{Q})$, and M_2. When a model matches M_2 and either $M_1(\mathbb{P})$ or $M_1(\mathbb{Q})$, it may not be possible to match the other first moment. Since the risk-neutral dynamics are typically estimated very precisely, this can lead to a difficulty matching $M_1(\mathbb{P})$ when M_2 is also matched.

3 Stochastic Volatility Term Structure Models

This section gives an overview of the stochastic volatility models that we consider. First, we establish a general factor time-series model with stochastic volatility that does not impose conditions for the absence of arbitrage. Within these models, arbitrary linear combinations of yields serve as instruments for volatility. An important consideration here is the admissibility conditions required to maintain a positive volatility process. Next, we show how no arbitrage conditions imply constraints on the general factor model. A key result that we show is that no arbitrage imposes that the volatility instrument is entirely determined by risk neutral expectations. Finally, we investigate further the links between volatility and the cross-sectional properties of the yield curve within the no arbitrage model. For simplicity, we focus in the main text on the case of a single volatility factor under a continuous time setup; modifications for discrete time processes and more technical details are described in Appendix B.

3.1 General admissibility conditions in latent factor models

We first review the conditions required for a well-defined positive volatility process within a multi-factor setting. Following Dai and Singleton (2000), hereafter DS, we refer to these conditions as admissibility conditions. Recall the N-factor $A_1(N)$ process of DS. This process has an N-dimensional state variable composed of a single volatility factor, V_t, and $(N-1)$ conditionally Gaussian state variables, X_t. The state variable $Z_t = (V_t, X_t')'$ follows the Itô diffusion

$$d\begin{bmatrix} V_t \\ X_t \end{bmatrix} = \mu_{Z,t}dt + \Sigma_{Z,t}dB_t^{\mathbb{P}}, \tag{5}$$

where

$$\mu_{Z,t} = \begin{bmatrix} K_{0V} \\ K_{0X} \end{bmatrix} + \begin{bmatrix} K_{1V} & K_{1VX} \\ K_{1XV} & K_{1X} \end{bmatrix} \begin{bmatrix} V_t \\ X_t \end{bmatrix}, \text{ and } \Sigma_{Z,t}\Sigma_{Z,t}' = \Sigma_{0Z} + \Sigma_{1Z}V_t, \tag{6}$$

and $B_t^{\mathbb{P}}$ is a standard N-dimensional Brownian motion under the historical measure, $\mathbb{P}$. Duffie, Filipovic, and Schachermayer (2003) show that this is the most general affine process on $\mathbb{R}_+ \times \mathbb{R}^{N-1}$.

In order to ensure that the volatility factor, V_t, remains positive, we need that when V_t is zero: (a) the expected change of V_t is non-negative, and (b) the volatility of V_t becomes zero. Otherwise there would be a positive probability that V_t will become negative. Imposing additionally the Feller condition for boundary non-attainment, our admissibility conditions are then

$$K_{1VX} = 0, \;\; \Sigma_{0Z,11} = 0, \;\; \text{and} \;\; K_{0V} \geq \frac{1}{2}\Sigma_{1Z,11}. \tag{7}$$

A consequence of these conditions is that volatility must follow an autonomous process under $\mathbb{P}$ since the conditional mean and variance of V_t depends only on V_t and not on X_t. We now show how to embed the $A_1(N)$ specification into generic term structure models where no arbitrage is not imposed and re-interpret these admissibility constraints in terms of conditions on the volatility instruments.

3.2 An $A_1(N)$ factor model without no arbitrage restrictions

We can extend the latent factor model of (5–6) to a factor model for yields by appending the factor equation

$$y_t = A_Z + B_Z Z_t, \tag{8}$$

where (A_Z, B_Z) are free matrices. Importantly, there are no cross-sectional restrictions that tie the loadings (A_Z, B_Z) together across the maturity spectrum. In this sense, this is a pure factor model without no arbitrage restrictions.

Given the parameters of the model, we can replace the unobservable state variable with observed yields through (8). Following Joslin, Singleton, and Zhu (2011), hereafter JSZ, we can identify the model by observing that equation (8) implies $\mathcal{P}_t \equiv W y_t = (WA_Z) + (WB_Z)Z_t$ for any given loading matrix W such that $\mathcal{P}_t$ is of the same size as Z_t. Assuming WB_Z is full rank,[11] this in turn allows us to replace the latent state variable Z_t with $\mathcal{P}_t$:

$$d\mathcal{P}_t = (K_0 + K_1 \mathcal{P}_t)dt + \sqrt{\Sigma_0 + \Sigma_1 V_t} dB_t^{\mathbb{P}}, \tag{9}$$

where we can write V_t (the first entry in Z_t) as a linear function of $\mathcal{P}_t$: $V_t = \alpha + \beta \cdot \mathcal{P}_t$.

Because the rotation from Z to $\mathcal{P}$ is affine, individual yields must be related to the yield factors $\mathcal{P}_t$ through:[12]

$$y_t = A + B\,\mathcal{P}_t. \tag{10}$$

The admissibility conditions (7) map into:

$$\beta' K_1 = c\beta', \tag{11}$$

where c is an arbitrary constant, and

$$\beta' \Sigma_0 \beta = 0, \text{ and } \beta' K_0 \geq \frac{1}{2}\beta' \Sigma_1 \beta. \tag{12}$$

We will denote the stochastic volatility model in (9–10) by $F_1(N)$. The model is parameterized by $\Theta_F \equiv (K_0, K_1, \Sigma_0, \Sigma_1, \alpha, \beta, A, B)$ which is subject to the conditions in (12). Our development shows that the $F_1(N)$ model is the most general factor model with an underlying affine $A_1(N)$ state variable.

[11] This is overidentifying. For details, see JSZ. In the current case, this would rule out unspanned stochastic volatility in the factor model. We extend our logic to the case of partially unspanned volatility in Section 8.

[12] To maintain internal consistency, we impose that $WA = 0$ and $WB = I_N$, as in JSZ. This guarantees that as we construct the yield factors by premultiplying W to the right hand side of the yield pricing equation (10), we exactly recover $\mathcal{P}_t$.

We will refer to the first admissibility condition in (11) as condition $A(\mathbb{P})$. This condition, needed so that V_t is an autonomous process under $\mathbb{P}$, can be restated as the requirement that β be a left eigenvector of K_1. With this requirement, choosing a β such that V_t matches yields volatility (M_2) is equivalent to imposing a certain left eigenvector on the time series feedback matrix K_1, which may hinder our ability to match the time series forecasts of bond yields ($M_1(\mathbb{P})$). When it is not possible to choose K_1 to match $M_1(\mathbb{P})$ and β to match M_2 in the presence of $A(\mathbb{P})$, a tension will arise. We refer to the tension between first and second moments as the difficulty to match $M_1(\mathbb{P})$ and M_2 in the presence of the constraint $A(\mathbb{P})$.

3.3 No arbitrage term structure models with stochastic volatility

The $A_1(N)$ no arbitrage model of DS represents a special case of the $F_1(N)$ model. That is, when one imposes additional constraints to the parameter vector Θ_F one will obtain a model consistent with no arbitrage. In this section, we first review the standard formulation of the $A_1(N)$ no arbitrage model. We then focus on the the effect of no arbitrage on the volatility instrument through the restriction it implies on the loadings parameter β.

The latent factor specification of the $A_1(N)$ model

We now consider affine short rate models which take a latent variable Z_t with dynamics given by (5–6) and append a short rate which is affine in a latent state variable. We consider the general market prices of risk of Cheridito, Filipovic, and Kimmel (2007). Joslin (2013a) shows that any such latent state term structure model can be drift normalized under $\mathbb{Q}$ so that we have the short rate equation

$$r_t = r_\infty + \rho_V V_t + \iota \cdot X_t, \tag{13}$$

where ι denotes a vector of ones, ρ_V is either +1 or -1, and the canonical risk-neutral dynamics of Z_t are given by

$$dZ_t = \left(\begin{bmatrix} K_{0V}^{\mathbb{Q}} \\ 0_{N-1\times 1} \end{bmatrix} + \begin{bmatrix} \lambda_V^{\mathbb{Q}} & 0_{1\times N-1} \\ 0_{N-1\times 1} & diag(\lambda_X^{\mathbb{Q}}) \end{bmatrix} Z_t \right) dt + \sqrt{\Sigma_{0Z} + \Sigma_{1Z} V_t} dB_t^{\mathbb{Q}}, \tag{14}$$

where $\lambda_X^{\mathbb{Q}}$ is ordered. To ensure the absence of arbitrage, we impose the Feller condition that $K_{0V}^{\mathbb{Q}} \geq \frac{1}{2}\Sigma_{1Z,11}$.

No arbitrage pricing then allows us to obtain the no arbitrage loadings that replace the unconstrained version of (8) in the $F_1(N)$ model with $y_t = A_Z^{\mathbb{Q}} + B_Z^{\mathbb{Q}} Z_t$ where $A_Z^{\mathbb{Q}}$ and $B_Z^{\mathbb{Q}}$ are dependent on the parameters underlying (13-14). From this, we again can rotate Z_t to $\mathcal{P}_t \equiv W y_t$ to obtain a yield pricing equation in terms of $\mathcal{P}_t$: $y_t = A + B\mathcal{P}_t$. This is a constrained version of the yield pricing equation for the $F_1(N)$ model in (10). In addition to the time series dynamics in (9), we also obtain the dynamics of $\mathcal{P}$ under $\mathbb{Q}$:

$$d\mathcal{P}_t = (K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} \mathcal{P}_t) dt + \sqrt{\Sigma_0 + \Sigma_1 V_t} dB_t^{\mathbb{Q}}, \tag{15}$$

with $V_t = \alpha + \beta \cdot \mathcal{P}_t$.

Compared to the $F_1(N)$ model, one clear distinction of the $A_1(N)$ model is the role of the $\mathbb{Q}$ dynamics (15) in determining yields loadings (A, B) and the volatility loadings β. We provide an in-depth discussion of this dependence below. We first explain the impact of the no arbitrage restrictions on the volatility loadings β. Next, we provide an intuitive illustration as to how the no arbitrage restrictions will give rise to an intimate relation between the yields loadings B and the volatility loadings β. This compares starkly with the $F_1(N)$ models for which B and β are completely independent.

Implications of the no arbitrage restrictions for the factor model

Ideally, we would like to characterize the no arbitrage model as restrictions on the parameter vector Θ_F in the $F_1(N)$ model. In JSZ, they were able to succinctly characterize the parameter restrictions of the no arbitrage model as a special case of the factor VAR model. In their case, essentially the main restriction was that the factor loadings (B) belongs to an N-parameter family characterized by the eigenvalues of the $\mathbb{Q}$ feedback matrix. In our current context of stochastic volatility models, such a simple characterization is not possible because changing the volatility parameters Σ_{1Z} affects not only the volatility structure but also the loadings $B_Z^{\mathbb{Q}}$.[13] This is because higher volatility implies higher convexity and thus higher bond prices or lower yields. The fact that Σ_{1Z} shows up both in volatility and in yields complicates a clean characterization of the restrictions on Θ_F that no arbitrage implies.

For this reason, we focus on a simpler but equally interesting question: what is the impact of the no arbitrage restrictions on the volatility loadings β?

Recall from the previous subsection that for an $F_1(N)$ model, the two main conditions on β are : (1) matching second moments (M_2); and (2) β must be a left eigenvector of the physical feedback matrix K_1 that matches the first moments under $\mathbb{P}$ ($M_1(\mathbb{P})$). Turning to the $A_1(N)$ model, these conditions are still applicable. Additionally, applying the admissibility conditions (7) to the risk-neutral dynamics in (15) results in a set of constraints analogous to (11):

$$\beta' K_1^{\mathbb{Q}} = c\beta', \tag{16}$$

for an arbitrary number c. We will refer to the condition in (16) for the no arbitrage model as the admissibility condition $A(\mathbb{Q})$. This implies a third condition on β for the no arbitrage model: β must be a left eigenvector of the risk neutral feedback matrix $K_1^{\mathbb{Q}}$ that matches the first moments under $\mathbb{Q}$ ($M_1(\mathbb{Q})$).

The impact of the no arbitrage restrictions on β depends on how strongly identifying the third condition is compared to the first two. Should $K_1^{\mathbb{Q}}$ be very precisely estimated from the data, the estimates of β for the $A_1(N)$ models are strongly influenced by $A(\mathbb{Q})$. Whence it is possible that β estimates are different across the $F_1(N)$ and $A_1(N)$ models. To anticipate our empirical results, we compare these restrictions in subsequent sections and indeed find that the admissibility condition $A(\mathbb{Q})$ (together with matching $M_1(\mathbb{Q})$ and M_2) is essentially

[13] In the Gaussian case, $B_Z^{\mathbb{Q}}$ is only dependent on the eigenvalues of the risk-neutral feedback matrix, and not on the volatility parameters.

the main restriction responsible for pinning down β in no arbitrage models whereas the direct tension between first and second moments implied by $A(\mathbb{P})$ has virtually no impact.

Why might $K_1^{\mathbb{Q}}$ be strongly pinned down in the data? Similar to JSZ, it can be shown that the no-arbitrage restriction on $K_1^{\mathbb{Q}}$ takes the following form:

$$K_1^{\mathbb{Q}} = (WB_Z^{\mathbb{Q}})diag(\lambda^{\mathbb{Q}})(WB_Z^{\mathbb{Q}})^{-1} \tag{17}$$

where $\lambda^{\mathbb{Q}} = (\lambda_V^{\mathbb{Q}}, \lambda_X^{\mathbb{Q}\prime})'$. This follows from the rotation from Z whose dynamics is given by (14) to $\mathcal{P}$. Additionally, observe that the loadings $B_Z^{\mathbb{Q}}$ depend only on $(\rho_V, \lambda^{\mathbb{Q}}, \Sigma_{1Z})$. Since ρ_V is a normalization factor, it can be ignored. Σ_{1Z} will affect the yield loadings through the Jensen effects which are typically small and will be dominated by variation in risk neutral expectations driven by $\lambda^{\mathbb{Q}}$. Thus $B_Z^{\mathbb{Q}}$ will be well approximated by loadings obtained when Σ_{1Z} is set to zeros. These can be viewed as loadings from a Gaussian term structure model which does not have a stochastic volatility effect. Up to this approximation, the risk-neutral feedback matrix is essentially a non-linear function of its eigenvalues, which are typically estimated with considerable precision (for example, see JSZ).[14] Combined, this implies that $K_1^{\mathbb{Q}}$ will be strongly identified in the data and thus β (up to scaling) is likely strongly affected by the no arbitrage restrictions due to $A(\mathbb{Q})$.

To relate to the results of JSZ, we make the above arguments relying on the approximation that convexity effects are negligible. It is important to note that we can make our argument more precise without resorting to approximations by a relatively more mechanical examination of the above steps. In particular, we show in Appendix A that the volatility instrument β is in fact, up to a constant, completely determined by the $(N-1)$ eigenvalues given in $\lambda_X^{\mathbb{Q}}$. Coupled with the observation that $\lambda_X^{\mathbb{Q}}$ is typically estimated with considerable precision, it is clear that the volatility instruments are heavily affected by the no arbitrage restrictions.

The relation between yield loadings and the volatility instrument

An alternative way of understanding the impact of the no arbitrage restrictions on the volatility instrument is through examining the linkage between yield loadings (B) and β. To begin, B and β are clearly independent for the $F_1(N)$ models since they are both free parameters. Intuitively, for these models the yields loadings B are obtained from purely cross-sectional information: regressions of yields on the pricing factors $\mathcal{P}$ whereas the volatility loadings β is obtained purely from the time series information. In contrast, in the context of an $A_1(N)$ model, both B and β are influenced by $K_1^{\mathbb{Q}}$. This common dependence on the risk-neutral feedback matrix forces a potentially tight linkage between these two components. For the sake of intuition, we consider below a simple example and show that for no arbitrage models there is indeed an intimate relationship between B and β.

[14] Intuitively, $\lambda^{\mathbb{Q}}$ governs the persistence of yield loadings along the *maturity* dimension. As shown by Joslin, Le, and Singleton (2012), the estimates of the loadings (obtained, for example, by projecting individual yields onto $\mathcal{P}_t$) are typically very smooth functions of yield maturities. This relative smoothness in turn should translate into small statistical errors associated with estimates of $\lambda^{\mathbb{Q}}$. This intuition is confirmed by examining the results of JSZ in which $\lambda^{\mathbb{Q}}$ is estimated with considerable precision.

Let's define the convexity-adjusted n-year forward rate on an one-year forward loan by:

$$f_t(n) = E_t^{\mathbb{Q}}[\int_{t+n}^{t+n+1} r_s ds]. \tag{18}$$

In the spirit of Collin-Dufresne, Goldstein, and Jones (2008) we can write the following one year ahead risk-neutral conditional expectation:

$$E_t^{\mathbb{Q}} \begin{pmatrix} V_{t+1} \\ f_{t+1}(0) \\ f_{t+1}(1) \end{pmatrix} = \text{constant} + \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 1 \\ a_2 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} V_t \\ f_t(0) \\ f_t(1) \end{pmatrix}. \tag{19}$$

The first row is due to the autonomous nature of V_t. The second row is the definition of the forward rate in (18) for $n = 1$. The last row is obtained from the fact that in a three factor affine model, (V_t, $f_t(0)$, $f_t(1)$) are informationally equivalent to the three underlying states at time t. From the last row and by applying the law of iterated expectation to (18), we have:

$$f_t(2) = \text{constant} + a_2 V_t + a_3 f_t(0) + a_4 f_t(1). \tag{20}$$

This equation may be solved to give V_t in terms of $f_t(0)$, $f_t(1)$, and $f_t(2)$. Furthermore, since (18) gives $f_{t+1}(2) = E_{t+1}^{\mathbb{Q}}[f_{t+2}(1)]$ we can use (19) and (20) to express $E_t^{\mathbb{Q}}[f_{t+1}(2)]$ in terms of $f_t(0)$, $f_t(1)$, and $f_t(2)$. Putting these together allows us to substitute V_t out from (19) and obtain

$$E_t^{\mathbb{Q}} \begin{pmatrix} f_{t+1}(0) \\ f_{t+1}(1) \\ f_{t+1}(2) \end{pmatrix} = \text{constant} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} f_t(0) \\ f_t(1) \\ f_t(2) \end{pmatrix}. \tag{21}$$

Simple calculations give $\alpha_1 = -a_1 a_3$, $\alpha_2 = a_3 - a_1 a_4$, and $\alpha_3 = a_4 + a_1$. It follows from the last row of (21) that:

$$f_t(3) = \text{constant} + \alpha_1 f_t(0) + \alpha_2 f_t(1) + \alpha_3 f_t(2). \tag{22}$$

Equation (22) reveals that if the forward rates can be empirically observed, the loadings α can in principle be pinned down simply by regressing $f_t(3)$ on $f_t(0)$, $f_t(1)$, and $f_t(2)$. Based on the mappings from (a_1, a_3, a_4) to α, it follows that the regression implied by (22) will also identify all the a coefficients, except for a_2. In the context of equation (20), it means that the volatility factor is tightly linked to the forward loadings, up to a translation and scaling effect. Since forwards and yields (and therefore yield portfolios) are simply rotated representations of one another, this implies a close relationship between the volatility instrument and yields loadings.

As is well known, yields and forwards at various maturities exhibit very high correlations. The R^2's obtained for cross-sectional regressions similar to (22) are typically close to 100% with pricing errors in the range of a few basis points. Therefore we expect the standard errors associated with α to be small and thus the volatility loadings β will be strongly identified from cross-sectional loadings.

Repeated iterations of the above steps allow us to write any forward rate $f_t(n)$ as a linear function of $(f(0)_t, f_t(1), f_t(2))$. Suppose that we use $J+1$ forwards in $(f_t(0), \ldots f_t(J))$ in estimation, then:

$$\begin{pmatrix} f_t(0) \\ f_t(1) \\ f_t(2) \\ f_t(3) \\ f_t(4) \\ \vdots \\ f_t(J) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ & g_4(\alpha) & \\ & \cdots & \\ & g_J(\alpha) & \end{pmatrix} \begin{pmatrix} f_t(0) \\ f_t(1) \\ f_t(2) \end{pmatrix}$$

where $(g_4, \ldots, g_J)$ represent the cross-sectional restrictions of no-arbitrage. This allows us to think of the no-arbitrage restrictions as having two facets. First, it imposes a cross-section to time series link through the fact that fixing α constrains what the volatility factor must look like, through a_3 and a_4. Second, it induces cross-sectional restrictions on the loadings $(g_4, \ldots g_J)$, just as is seen with pure Gaussian term structure models.

4 Evaluating the Admissibility Restrictions

We have seen in Section 3 that in order to have a well-defined admissible volatility process, we must have both $A(\mathbb{P})$ and $A(\mathbb{Q})$ which can be restated as that β must be a common left eigenvector of the feedback matrices under $\mathbb{P}$ and $\mathbb{Q}$. These admissibility restrictions are helpful in providing guidance on potential volatility instruments. For example, although level is known to be related to volatility, it is unlikely to be an admissible instrument for volatility by itself. To see this, recall the well-known result (for example Campbell and Shiller (1991)) that the slope of the yield curve predicts future changes in the level of interest rates. Up to the associated uncertainty of such statistical evidence, this suggests that the slope of the yield curve predicts the level and thus also that the level of interest rates is not an autonomous process.

We evaluate empirically how helpful each of the admissibility restrictions can be in identifying the potential volatility instrument which in turn depends on the accuracy with which the feedback matrices can be estimated. For example, if the physical (risk-neutral) feedback matrix is strongly identified in the data, then the condition $A(\mathbb{P})$ $(A(\mathbb{Q}))$ must provide helpful identifying information about β. As will be seen, our assessments are relatively robust to the extent that we do not have to actually estimate the term structure models, nor do we require that M_2 be matched. Following Joslin, Le, and Singleton (2012) (hereafter JLS), we use the monthly unsmoothed Fama Bliss zero yields with eleven maturities: 6–month, one- out to ten-year. We start our sample in January 1973, due to the sparseness of longer maturity yields prior to this period, and end in December 2007 to ensure our results are not influenced by the financial crisis.

We note that the affine dynamics for $\mathcal{P}$ in (9) implies that the one month ahead conditional

expectation of $\mathcal{P}_{t+\Delta}$ is affine in $\mathcal{P}_t$:

$$E_t[\mathcal{P}_{t+\Delta}] = \text{constant} + e^{K_1\Delta}\mathcal{P}_t \tag{23}$$

where $\Delta = 1/12$. Thus $\mathcal{P}_t$, even when sampled monthly, follows a first order VAR. Importantly, we can show that any left eigenvector of K_1 must also be a left eigenvector of the one-month ahead feedback matrix $e^{K_1\Delta}$, denoted by $K_{1,\Delta}$.[15] In other words, the set of left eigenvectors of the instantaneous feedback matrix K_1 and the one-month ahead feedback matrix $K_{1,\Delta}$ must be identical. As a result, we can equivalently restate $A(\mathbb{P})$ as the requirement that the volatility loading β be a left eigenvector of $K_{1,\Delta}$. Since our data are sampled at the monthly interval, it is more convenient for us to focus on $K_{1,\Delta}$ in our empirical analysis.

Similarly, the affine dynamics in (15) under $\mathbb{Q}$ also implies a first order VAR for $\mathcal{P}_t$ sampled at the monthly frequency:

$$E_t^{\mathbb{Q}}[\mathcal{P}_{t+\Delta}] = constant + \underbrace{e^{K_1^{\mathbb{Q}}\Delta}}_{K_{1,\Delta}^{\mathbb{Q}}}\mathcal{P}_t. \tag{24}$$

Applying similar logic, we can again restate $A(\mathbb{Q})$ as the requirement that the volatility loading β be a left eigenvector of the one-month ahead risk-neutral feedback matrix $K_{1,\Delta}^{\mathbb{Q}}$.

It is worth noting that for small Δ, $K_{1,\Delta}^{\mathbb{Q}} \approx I + \Delta K_1^{\mathbb{Q}}$. So in some sense, we can view $K_1^{\mathbb{Q}}$ and $K_{1,\Delta}^{\mathbb{Q}}$ interchangeably. Importantly though, as the arguments above illustrate, our results do not rely on this approximation.

4.1 Admissibility restrictions under $\mathbb{P}$

We first consider the restriction $A(\mathbb{P})$ which is present in both the $F_1(N)$ and $A_1(N)$ models. This restriction guarantees that V_t is an autonomous process, which in turn is necessary for volatility to be a positive process under $\mathbb{P}$. This requires the volatility instrument, β, be a left eigenvector of the one-month ahead physical feedback matrix $K_{1,\Delta}$. To the extent that the conditional mean is strongly identified by the time-series, this condition will pin down the admissible volatility instruments up to a sign choice and the choice of which of the N left eigenvectors instruments volatility. However, in general even with a moderately long time series, such as our thirty five year sample, inferences on the conditional means are not very precise.

To gauge how strongly identified the volatility instrument is from the autonomy requirement under $\mathbb{P}$, we implement the following exercise. First we estimate an unconstrained VAR on the first three principal factors, $\mathcal{P}_t$. Ignoring the intercepts, the estimates for our sample

[15] To see this, assume that β is a left eigenvector of K_1 with a corresponding eigenvalue c. Applying the definition of left eigenvector, $\beta' K_1 = c\beta'$, repeatedly, it follows that $\beta' K_1^n = c^n \beta'$ or β is also a left eigenvector of K_1^n for any n. Substitute these into $e^{K_1\Delta} = \sum_{n=0}^{\infty} K_1^n \Delta^n / n!$, it implies that $\beta' e^{K_1\Delta} = e^{c\Delta}\beta'$. Thus β is a left eigenvector of $e^{K_1\Delta}$ with the corresponding eigenvalue $e^{c\Delta}$.

period are:

$$\mathcal{P}_{t+\Delta} = \text{constant} + \underbrace{\begin{pmatrix} 0.9902 & -0.0092 & -0.0472 \\ 0.0097 & 0.9548 & -0.0802 \\ -0.0021 & 0.0096 & 0.7991 \end{pmatrix}}_{K_{1,\Delta}} \mathcal{P}_t + noise. \tag{25}$$

Then, for *each* potential volatility instrument $\beta \cdot \mathcal{P}_t$ (as β roaming over all possible choices), we re-estimate the VAR under the constraint that β is a left eigenvector of $K_{1,\Delta}$. The VAR is easily estimated under this constraint after a change of variables so that the eigenvector constraint becomes a zero constraint (compare the constraints in (7) and (12)). We then conduct a likelihood ratio test of the unconstrained versus the constrained alternative and compute the associated probability value (p-value). A p-value close to one indicates that the evidence is consistent with such an instrument being consistent with $A(\mathbb{P})$ while a p-value close to zero indicates contradicting evidence.[16] In conducting this experiment, we do not force $\beta \cdot \mathcal{P}_t$ to forecast volatility nor is β required to satisfy $A(\mathbb{Q})$. In this sense, this exercise is informative about the contribution of $A(\mathbb{P})$ in shaping the volatility instrument independent of both $A(\mathbb{Q})$ and the requirement that M_2 be matched.

Since $\beta \cdot \mathcal{P}_t$ and its scaled version, $c\beta \cdot \mathcal{P}_t$, for any constant c, effectively give the same volatility factor (and hence deliver the same p-values in our exercise), we scale so that all elements of β sum up to one (the loading on PC1 $\beta(1) = 1 - \beta(2) - \beta(3)$). We plot the p-values against the corresponding pairs of loadings on PC2 and PC3 in Figure 2. For ease of presentation, in this graph the three PCs are scaled to have in-sample variances of one.

We see that there are three peaks which correspond to the three left eigenvectors of the maximum likelihood estimate of $K_{1,\Delta}$. When β is equal to one of these left eigenvectors (up to scaling), the likelihood ratio test statistic must be zero and hence the corresponding p-value must be one, by construction. As our intuition suggests, many, though not all, instruments appear to potentially satisfy $A(\mathbb{P})$ according to the metric that we are considering. Thus we conclude that the admissibility requirement under the $\mathbb{P}$ measure in general still leaves a great deal of flexibility in forming the volatility instrument.

4.2 Admissibility restrictions under $\mathbb{Q}$

Turning to $A(\mathbb{Q})$, to have a clean comparison, it is ideal if we can implement the same regression approach applied to $A(\mathbb{P})$ in the previous exercise. That is, we first run an unconstrained regression using the $\mathbb{Q}$ forecasts:

$$E_t^{\mathbb{Q}}[\mathcal{P}_{t+\Delta}] = constant + K_{1,\Delta}^{\mathbb{Q}} \mathcal{P}_t + noise \tag{26}$$

to obtain an estimate of $K_{1,\Delta}^{\mathbb{Q}}$. An important difference here with the $\mathbb{P}$ case in (25) is that we now use $E_t^{\mathbb{Q}}[\mathcal{P}_{t+\Delta}]$ instead of $\mathcal{P}_{t+1}$ on the left hand side in the regression. Next, for

[16] We view this test as an approximation since it assumes volatility of the residuals is constant. However, computations of p-values, accounting for heteroskedasticity of the errors, deliver very similar results.

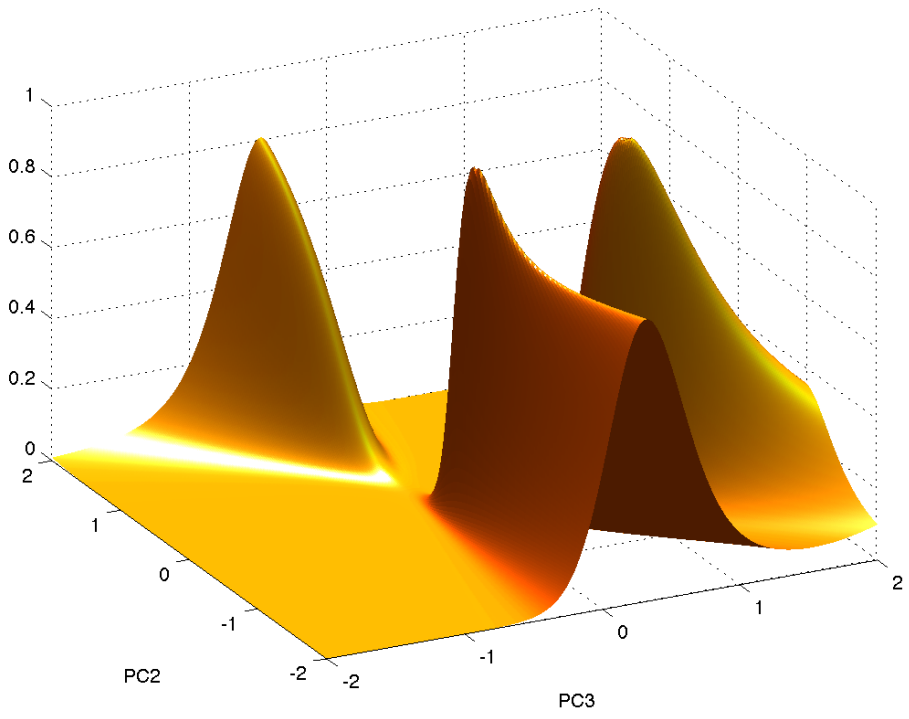


Figure 2: Likelihood Ratio Tests of the Autonomy Restriction under $\mathbb{P}$. This figure reports the p-values of the likelihood ratio test of whether a particular linear combination of yields, $\beta \cdot \mathcal{P}_t$, is autonomous under $\mathbb{P}$, plotted against the loadings of PC2 and PC3. The loading of PC1 is one minus the loadings on PC2 and PC3 ($\beta(1) = 1 - \beta(2) - \beta(3)$). PC1, PC2, and PC3 are scaled to have in-sample variances of one.

each potential volatility instrument $\beta \cdot \mathcal{P}_t$, we re-estimate the regression in (26) under the constraint that β is a left eigenvector of $K^{\mathbb{Q}}_{1,\Delta}$. As is seen in the previous exercise, the resulting likelihood ratios reveal whether or not the volatility instrument considered is consistent with the admissibility constraint $A(\mathbb{Q})$.

Although we do not strictly observe the risk neutral forecasts $E^{\mathbb{Q}}_t[\mathcal{P}_{t+\Delta}]$ for stochastic volatility models due to the presence of convexity effects, we use a model-free approach to obtain very good approximation. The insight again is that risk-neutral expectations are, up to convexity, observed as forward rates. The n-year forward rate that begins in one month, $f^{\Delta,n}_t = \frac{1}{n}((n+\Delta)y_{n+\Delta,t} - \Delta r_t)$ is, up to convexity effects:

$$f^{\Delta,n}_t \approx E^{\mathbb{Q}}_t[y_{n,t+\Delta}] \tag{27}$$

where $y_{n,t}$ denotes n-year zero yield observed at time t. Thus we can use (27) to approximate $E^{\mathbb{Q}}_t[y_{n,t+\Delta}]$ whereby we simply ignore any convexity terms. This approximation is reasonable

for two reasons. First, Jensen terms are typically small. Second, notice that since our primary interest is not in the level of expected-risk neutral changes but in their variation (as captured by $K^{\mathbb{Q}}_{1,\Delta}$), it is only changes in stochastic convexity effects that will violate this approximation. Thus to the extent that *changes* in convexity effects are small this approximation will be valid for inference of $K^{\mathbb{Q}}_{1,\Delta}$.

Using this method, we extract observations on $E^{\mathbb{Q}}_t[y_{n,t+\Delta}]$ from forward rates which we can then convert into estimates of $E^{\mathbb{Q}}_t[\mathcal{P}_{t+\Delta}]$ using the weighting matrix W. We denote this approximation of $E^{\mathbb{Q}}_t[\mathcal{P}_{t+\Delta}]$ by $\mathcal{P}^f_t$. Whence regression (26) translates into:

$$\mathcal{P}^f_t = constant + K^{\mathbb{Q}}_{1,\Delta}\mathcal{P}_t + noise. \tag{28}$$

Regression (28) draws a nice analogy to the time series VAR(1) of (25) that we use in examining $A(\mathbb{P})$. Importantly, as this regression can be implemented completely independently, abstracting from any time series considerations, it serves as a stand-alone assessment of $A(\mathbb{Q})$, up to the validity of our convexity approximation approach. Notably, (28) makes clear the (essentially) contemporaneous nature of the estimation of $K^{\mathbb{Q}}_{1,\Delta}$. Since $\mathcal{P}_t$ explains virtually all contemporaneous yields and forwards (and thus portfolios of forwards such as $\mathcal{P}^f_t$), the R^2's of (28) are likely much higher than those for the time series VAR(1) at the monthly frequency. Therefore we expect much stronger identification for $K^{\mathbb{Q}}_{1,\Delta}$. Intuitively, although we observe only a single time series under the historical measure with which to draw inferences, we observe repeated term structures of risk-neutral expectations every month and this allows us to draw much more precise inferences.

Figure 3 plots the p-values for this test of the restrictions of various instruments to be autonomous under $\mathbb{Q}$. In stark contrast to Figure 2 and in accordance with our intuition, we see that the risk-neutral measure provides very strong evidence for which instruments are able to be valid volatility instruments. Most potential volatility instruments are strongly ruled out with p-values essentially at zero. Thus, our results here suggest that were it only up to $A(\mathbb{P})$ and $A(\mathbb{Q})$ to decide which volatility instrument to use, the latter would almost surely be the dominant force, with the remaining degrees of freedom being the sign choice and choosing which of the N left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$ is the volatility instrument. This evidence suggests that the no arbitrage restrictions can potentially have very strong impact in shaping volatility choices.

Left open by the model-free nature of our analysis in this section is, among other things, the possibility that the defining property of the volatility factor (β should match M_2) can be powerful enough that it might dominate $A(\mathbb{Q})$ at identifying potential volatility instruments. We take up an in depth examination of this possibility in the next section.

5 Comparison of Gaussian and Stochastic Volatility Models

To understand the contribution of matching M_2 on the identification of the volatility loadings β, we estimate and compare the (Gaussian) $A_0(N)$ models with stochastic volatility models.

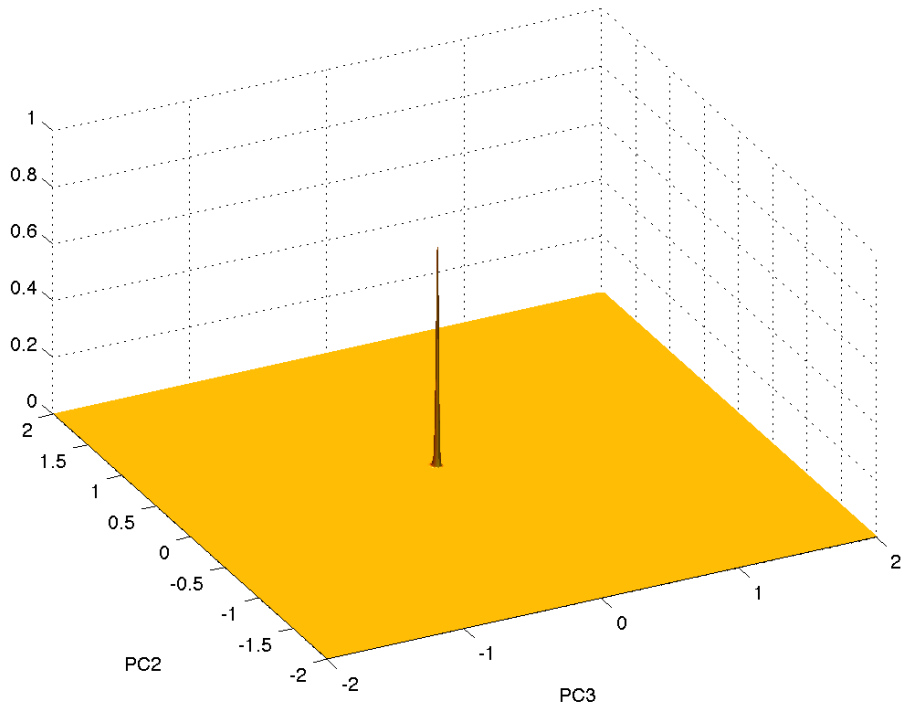


Figure 3: Likelihood Ratio Tests of the Autonomy Restriction under $\mathbb{Q}$. This figure reports the p-values of the likelihood ratio test of whether a particular linear combination of yields, $\beta \cdot \mathcal{P}_t$, is autonomous under $\mathbb{Q}$, plotted against the loadings of PC2 and PC3. The loading of PC1 is one minus the loadings on PC2 and PC3 ($\beta(1) = 1 - \beta(2) - \beta(3)$). PC1, PC2, and PC3 are scaled to have in-sample variances of one.

	N = 4				N = 3		
$A_0(N)$	0.998	0.027	0.032	0.014	0.997	0.028	0.025
	-0.007	0.957	-0.128	-0.042	-0.003	0.954	-0.098
	-0.010	0.006	0.895	-0.080	-0.005	-0.002	0.928
	-0.009	-0.009	-0.085	1.007			
$A_1(N)$	0.999	0.027	0.030	0.013	0.998	0.028	0.024
	-0.005	0.959	-0.123	-0.037	-0.002	0.955	-0.097
	-0.010	0.006	0.902	-0.075	-0.005	-0.000	0.931
	-0.006	-0.007	-0.079	1.018			
$A_2(N)$	0.998	0.028	0.031	0.013	0.997	0.029	0.025
	-0.005	0.956	-0.125	-0.040	-0.002	0.954	-0.099
	-0.009	0.003	0.899	-0.077	-0.005	-0.002	0.929
	-0.005	-0.012	-0.080	1.010			
Regression	0.998	0.027	0.029	0.014	0.997	0.027	0.025
	-0.008	0.957	-0.118	-0.041	-0.003	0.958	-0.098
	-0.011	0.005	0.905	-0.077	-0.006	0.007	0.925
	-0.016	-0.006	-0.065	0.987			

Table 1: $K^{\mathbb{Q}}_{1,\Delta}$ Estimates.

Clearly, matching M_2 is relevant only in the latter and not the former. Since the $A_0(N)$ models are affine models, the one month ahead conditional expectation of yields portfolios $\mathcal{P}$ also take an affine form. Thus for both Gaussian and stochastic volatility models, we can write: $E^{\mathbb{Q}}_t[\mathcal{P}_{t+\Delta}] = \text{constant} + K^{\mathbb{Q}}_{1,\Delta}\mathcal{P}_t$ under the risk neutral measure. Of particular interest is the estimates of the monthly risk-neutral feedback matrix, $K^{\mathbb{Q}}_{1,\Delta}$, implied by these models. As we will show in this section, estimates of $K^{\mathbb{Q}}_{1,\Delta}$ are highly similar across these models. This suggests that the role of stochastic volatility (matching M_2) is inconsequential for the estimation of $K^{\mathbb{Q}}_{1,\Delta}$. Thus identifying volatility instrument (β) is simply limited to making the choice of which left eigenvector of $K^{\mathbb{Q}}_{1,\Delta}$ and its sign can best match M_2. We use the same dataset as in the preceding section and note that all of our results remain fully robust for a shortened sample period that excludes the Fed experiment regime.

5.1 Comparison of $K^{\mathbb{Q}}_{1,\Delta}$ estimates

We estimate $A_M(N)$ models, with $M = 0, 1, 2$ and $N = 3, 4$, and then rotate the state variables into low order yield PCs. For estimation, we assume these PCs are priced perfectly while higher order PCs are observed with i.i.d. errors. JLS show that this assumption is innocuous as it is likely to deliver estimates close to those obtained by Kalman filtering where all yields portfolios are observed with errors. Estimation details and full parameter estimates are deferred to Appendix C.

Table 1 reports the estimates of $K^{\mathbb{Q}}_{1,\Delta}$ implied by these models. Recall the defining property of $K^{\mathbb{Q}}_{1,\Delta}$ given by equation (26) in which $K^{\mathbb{Q}}_{1,\Delta}$ is informative about how $\mathcal{P}_t$ forecasts $\mathcal{P}_{t+\Delta}$ under the risk neutral measure. Since for each N, $\mathcal{P}_t$ is characterized by the same loading matrix W (that corresponds to the first N PCs of bond yields) across all models, it follows that $K^{\mathbb{Q}}_{1,\Delta}$ estimates are directly comparable across all models with the same number of factors N. Focusing first on the two models $A_0(3)$ and $A_1(3)$, the two estimates of $K^{\mathbb{Q}}_{1,\Delta}$ are strikingly close: most entries are essentially identical up to the third decimal place. This evidence indicates that the identification by the cross-sectional information (and possibly other moments shared between the $A_0(3)$ and $A_1(3)$ models) for the parameter $K^{\mathbb{Q}}_{1,\Delta}$ seems overwhelmingly stronger than the restrictions coming from matching M_2. Enriching the volatility structure to $M = 2$ does not overturn this observation: the $K^{\mathbb{Q}}_{1,\Delta}$ estimate implied by the $A_2(3)$ model remains essentially identical. Additionally, changing the number of factors to $N = 4$ (results also reported Table 1) or $N = 2$ (results not reported) does not alter our observation.

We have argued that variation in the one month ahead risk-neutral expectations, as determined by $K^{\mathbb{Q}}_{1,\Delta}$, is well approximated by the regression based estimate of (28). This estimate can be further improved by simple steps that take into account the affine structure of bond yields. Specifically up to convexity effects, the affine structure of bond yields implies that:

$$\tilde{B}_{n+\Delta} = K^{\mathbb{Q}}_{1,\Delta}\tilde{B}_n + \tilde{B}_\Delta$$

where $\tilde{B}_n$ denotes the unannualized loadings of n-year zero yields on $\mathcal{P}_t$. This suggests we can recover $K^{\mathbb{Q}}_{1,\Delta}$ in two steps. First, we project yields of all maturities onto the states $\mathcal{P}_t$ to recover the loadings $\tilde{B}_n$.[17] Second, an estimate of $K^{\mathbb{Q}}_{1,\Delta}$ is obtained by projecting $\tilde{B}_{n+\Delta} - \tilde{B}_\Delta$ onto $\tilde{B}_n$ (allowing for no intercepts).[18] As can be viewed from the last panel of Table 1, this model free estimate of $K^{\mathbb{Q}}_{1,\Delta}$ come strikingly close to estimates obtained from the no arbitrage models. This evidence suggests that the cross-sectional information *alone* is sufficient to pin down the risk-neutral feedback matrix, and this identification is so strong that information from other constraints imposed by the models seems irrelevant.

Given the estimates of $A_M(N)$ models, we are able to confirm that the convexity effects on yield loadings are negligible. Specifically, holding N fixed, varying M, and thereby varying the degree of convexity effects due to the presence of stochastic volatility, is completely inconsequential for the yield loadings implied by different models. Graphs (not reported) of yield loadings on $\mathcal{P}_t$ plotted against the corresponding maturities (up to ten years) implied by $A_0(N)$, $A_1(N)$, and $A_2(N)$ are virtually indistinguishable.

The observed invariance property of $K^{\mathbb{Q}}_{1,\Delta}$ estimates has a number of implications. First, as stated previously, this allows us to pin down the potential volatility instruments using the cross-section of yields due to the admissibility constraint. Essentially the volatility instrument is free in terms of the sign but must be one of the left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$ which can be

[17] To obtain yields for the full range of maturities from the small set of maturities used in estimation, we can use simple interpolation techniques such as the constant forward bootstrap or simply a cubic spline.

[18] de los Rios (2013) develops a similar regression-based approach to obtain estimates of $K^{\mathbb{Q}}_{1,\Delta}$.

computed accurately from either the cross-sectional regression or from estimation of the $A_0(N)$ model which has constant volatility and can be estimated quite quickly as shown in JSZ.

This observation also shows that, in some regards, the estimation of the no arbitrage $A_1(N)$ model is more tractable than estimate of the $\mathrm{F}_1(N)$ model. In the case of the Gaussian models the opposite holds: the factor model is trivial to estimate as it amounts to a set of ordinary least squares regressions while the no arbitrage model is slightly more difficult to estimate due to the non-linear constraints in the factor loadings. In the stochastic volatility models, the admissibility conditions require a number of non-linear constraints in order to ensure that volatility remains positive. The no arbitrage model essentially determines the volatility instrument up to sign and choice of eigenvector. This actually simplifies the estimation since it reduces the set of non-linear constraints that need to be imposed.

The observation that $K^{\mathbb{Q}}_{1,\Delta}$ estimates are nearly invariant across Gaussian and stochastic volatility models leads us to the surprising conclusion that the $A_0(N)$ model with *constant volatility* allows us to essentially identify (up to choice of which eigenvector) the source of stochastic volatility in the $A_1(N)$ model. We provide an illustration of this point in the next subsection.

5.2 Volatility information revealed by the Gaussian model

Despite the similarity, the estimates of $K^{\mathbb{Q}}_{1,\Delta}$ reported in Table 1 still exhibit slight numerical differences. It is possible these small numerical differences might become more significant in terms of the left eigenvectors and thus among model implied volatility instruments. To show that this is not the case, we carry out the following exercise. Starting with the $K^{\mathbb{Q}}_{1,\Delta}$ estimate by the $A_0(3)$ model, we form three potential volatility instruments from the three left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$ and then pick out the instrument with most predictive content for volatility. Specifically, we first project the level factor, $\mathcal{P}_{1,t+\Delta}$, onto $\mathcal{P}_t$ to obtain the forecast residuals and then choose the volatility candidate with most predictive content for the squared residuals. This way, from the $A_0(3)$ model, we can have a "guess" for what the volatility instrument of the $A_1(3)$ model looks like even before we actually estimate the $A_1(3)$ model. Finally, we compare this "guess" to the actual volatility instrument implied by the $A_1(3)$ model.

Table 2 reports the adjusted R^2 statistics (in percentage) of regressions in which each potential volatility instrument is used to predict the squared residuals of the level factor. Evidently, one of the instruments clearly dominates the others at all forecasting horizons from one to twelve months. Comparing this dominant instrument to the actual volatility factor of the $A_1(3)$ model results in a striking correlation of one. To see this more visually, we plot these two volatility instruments, normalized to have the same scaling and intercepts,[19] in Figure 4. Clearly, the $A_0(3)$'s "guess" is very accurate as the two graphs are right on top of one another.

[19]Specifically, in constructing both volatility instruments, we drop the intercepts and scale the loading on the level factor ($\beta(1)$) to one.

Horizon	Instrument 1	Instrument 2	Instrument 3
1	9.35	-0.00	0.58
2	8.84	-0.21	-0.16
3	8.66	0.92	2.17
4	7.32	0.96	2.55
5	6.51	0.50	1.84
6	6.04	-0.24	0.12
7	5.28	-0.07	0.57
8	4.71	-0.24	0.01
9	4.87	0.33	1.26
10	4.59	0.51	1.66
11	4.27	0.51	1.77
12	4.21	0.03	0.95

Table 2: R^2 (in percentage) predicting squared residuals in forecasting the level factor by the three potential volatility instruments implied by the $A_0(3)$ model. Instruments 1, 2, 3 are formed from the left eigenvectors of the $K^{\mathbb{Q}}_{1,\Delta}$ matrix, corresponding to the eigenvalues ordered from highest to lowest.

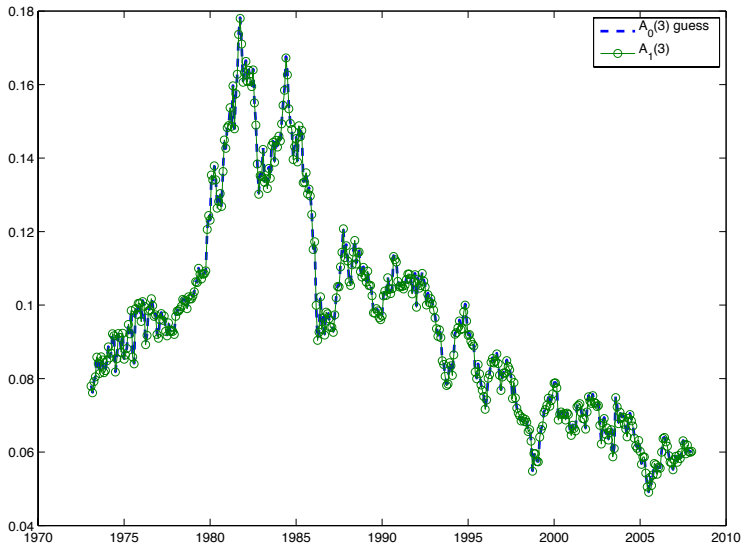


Figure 4: Volatility instrument "guessed" by the $A_0(3)$ model and the actual volatility factor implied by the $A_1(3)$ model. The volatility instrument is normalized as $\beta \cdot \mathcal{P}_t$ where $\beta(1)$ is scaled to one.

This exercise and the content of the previous subsection clearly reveal the respective roles of the cross-sectional and time series information in shaping the choice of volatility instrument in an $A_1(3)$ model. The cross-sectional information pins down the risk-neutral feedback matrix $K^{\mathbb{Q}}_{1,\Delta}$. The identification seems so strong that time series constraints from matching $M_1(\mathbb{P})$ and M_2 appear inconsequential. Regardless of whether the time series constraints are applied (in the $A_1(3)$ model) or not (in the $A_0(3)$ model), the estimates of $K^{\mathbb{Q}}_{1,\Delta}$ seem largely unaffected. The precise estimate of $K^{\mathbb{Q}}_{1,\Delta}$ together with $A(\mathbb{Q})$ dramatically reduces the choice of potential volatility instruments from an uncountably infinite set to a discrete choice among the N left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$. This is the very sense in which a constant volatility model such as the $A_0(N)$ model can reveal volatility information of stochastic volatility models. The $A_0(N)$ model pins the optimal volatility instrument to be one of the N left eigenvectors, but it does not determine exactly which one. It is the role of the time series constraints in picking the left eigenvector (and its sign) that best matches $M_1(\mathbb{P})$ and M_2.

6 No Arbitrage Restrictions

In this section, we show that the clear distinction of the roles of cross-section and time series information in determining the volatility of no arbitrage models can have important implications for dynamic term structure models. Specifically, we reconsider the puzzling result of Dai and Singleton (2002). They show that while Gaussian models are able to replicate the deviations from the expectation hypothesis found in the data, affine term structure models with stochastic volatility are unable to match the patterns found in the data. This failure may potentially be due to the tension between first and second moments. However, we show that the stochastic volatility factor model (where the first and second moment tension still applies) is able to match deviations from the expectations hypothesis. This demonstrates that the tension created by also matching $M_1(\mathbb{Q})$ in addition to $M_1(\mathbb{P})$and M_2 is what drives the failures of the stochastic volatility models demonstrated by Dai and Singleton (2002).

These results, together with our previous results, show that recent results about the irrelevancy of no arbitrage restrictions in Gaussian models do not extend to affine models with stochastic volatility. For example, Duffee (2011), Joslin, Singleton, and Zhu (2011), and Joslin, Le, and Singleton (2012) all show that no arbitrage is nearly irrelevant in Gaussian dynamic term structure models on a number of dimensions. In contract, for the case of stochastic volatility models, the no arbitrage constraints on the factor model has material effects for both first and second moments.

6.1 Expectation hypothesis

A generic property of arbitrage-free dynamic term structure models is that risk-premium adjusted expected changes in bond yields are proportional to the slope of the yield curve. Under the expectation hypothesis (EH), risk premiums are constant. This implies that the

coefficients ϕ_n in the projections

$$Proj\left[y_{n-\Delta,t+\Delta} - y_{n,t} | y_{n,t} - y_{\Delta,t}\right] = \alpha_n + \phi_n \left(\frac{y_{n,t} - y_{\Delta,t}}{n - \Delta}\right), \text{ for all } n > \Delta, \tag{29}$$

should be uniformly ones under the EH. Campbell and Shiller (1991) shows robust evidence that ϕ_n's are significantly different from one and become increasingly negative for large n's. This puzzling pattern of ϕ_n's, which can be observed in Figure 1 for our sample period, has become one of the most studied empirical phenomena for the last twenty years.

Dai and Singleton (2002) show that constant volatility models are not "puzzled" by this pattern and that the population coefficients ϕ_n implied by estimated $A_0(N)$ models very closely match their data counterparts. However, Dai and Singleton (2002) show a stark contrast for the canonical models $A_M(N)$ models with $M > 0$ with stochastic volatility. Here they find ϕ_n's typically stay close to the unit line, thereby counter-factually implying that the EH nearly holds.

What is behind the difference in performances of the Gaussian and stochastic volatility models? To begin, it is worth noting that the loadings ϕ_n's for all affine models (with or without no arbitrage restrictions) can be written as:

$$\phi_n = (n - \Delta)\frac{(B_{n-\Delta}K_{1,\Delta} - B_n)\Sigma(B_n - B_\Delta)'}{(B_n - B_\Delta)\Sigma(B_n - B_\Delta)'}. \tag{30}$$

where Σ denotes the unconditional covariance matrix of the time series innovations and B_n the loadings of the n-period yield $y_{n,t}$ on the principal components of yields $\mathcal{P}_t$. As noted earlier, the loadings B's are essentially identical across models with and without stochastic volatility.[20] Furthermore, the covariance matrix Σ appears in both the numerator and denominator of (30), thus its impact on ϕ_n is greatly dampened due to cancellation. This essentially leaves the one-month ahead physical feedback matrix $K_{1,\Delta}$ as the natural focus in explaining the differences in ϕ_n's across the constant and stochastic volatility models.

One of the main findings of JSZ in the context of the Gaussian models is that no arbitrage restrictions are irrelevant for models' forecasting performance. Equivalently, estimates of the one-month ahead physical feedback matrix $K_{1,\Delta}$ from $A_0(N)$ models are exactly identical to those obtained from OLS regressions of $\mathcal{P}_{t+\Delta}$ on $\mathcal{P}_t$ and, thus, completely unaffected by no arbitrage restrictions. Turning to the $A_1(N)$ models, the concurrent presence of $A(\mathbb{P})$ and $A(\mathbb{Q})$ builds a strong link between the one-month ahead physical and risk neutral feedback matrices: $K_{1,\Delta}$ and $K^{\mathbb{Q}}_{1,\Delta}$ must share one common left eigenvector. To the extent that $K^{\mathbb{Q}}_{1,\Delta}$ is very strongly pinned down by the cross-section information, it is likely to force the physical feedback matrix $K_{1,\Delta}$ to accept one of the N left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$ as one of its own. Due to this coupling of $K_{1,\Delta}$ and $K^{\mathbb{Q}}_{1,\Delta}$, the estimate of $K_{1,\Delta}$ from the $A_1(N)$ model is likely strongly influenced by the no arbitrage restrictions and thus can be quite different from its OLS counter-part.

[20] In fact, these loadings are very close to those obtained from OLS regressions of yields of individual maturities onto the pricing factors $\mathcal{P}_t$.

	N = 4				N = 3		
$A_0(N)$	0.990	-0.009	-0.047	-0.017	0.990	-0.009	-0.047
	0.010	0.955	-0.080	-0.032	0.010	0.955	-0.080
	-0.002	0.010	0.799	0.030	-0.002	0.010	0.799
	0.000	0.012	-0.012	0.627			
$A_1(N)$	0.990	0.015	0.002	-0.034	0.993	0.011	-0.035
	0.004	0.975	-0.080	-0.099	0.006	0.973	-0.100
	0.000	0.002	0.835	0.013	-0.001	-0.009	0.823
	-0.006	0.001	-0.031	0.703			
$A_2(N)$	0.987	0.017	0.035	-0.004	0.992	0.015	0.009
	-0.002	0.958	-0.083	-0.106	0.004	0.974	-0.075
	0.002	0.013	0.870	0.052	0.013	0.011	0.902
	0.001	0.020	-0.025	0.729			
$F_1(N)$	0.991	-0.013	-0.093	-0.010	0.997	-0.018	-0.081
	0.006	0.962	-0.054	-0.045	0.002	0.965	-0.052
	0.009	-0.012	0.867	0.048	0.005	-0.008	0.830
	-0.009	0.022	0.003	0.715			
$F_2(N)$	0.994	-0.013	-0.086	0.028	0.998	-0.018	-0.056
	0.005	0.962	-0.067	-0.070	0.005	0.965	-0.010
	0.005	-0.012	0.876	0.032	0.004	-0.007	0.828
	-0.003	-0.011	-0.027	0.702			

Table 3: $K_{1,\Delta}$ Estimates

Table 3 reports estimates of $K_{1,\Delta}$ for the $A_M(N)$ models with $M = 0, 1, 2$ and $N = 3, 4$. The sample period is 1973 through 2007 and we note, again, that all of our results for a shortened sample period that excludes the Fed experiment regime remain qualitatively similar. Comparing the $A_0(N)$ and $A_1(N)$ models reveals one interesting difference: for both $N = 3$ and $N = 4$, the (1,2) entry of the feedback matrix, which governs how slope this period forecasts level next month, is negative for the $A_0(N)$ model but positive for the $A_1(N)$ model. A negative value for this entry means that higher slope leads to lower level and thus higher return in the future whereas the opposite is true for a positive entry. As is well-known, the Campbell and Shiller (1991) regression in (29) is equivalent to one in which future bonds' excess returns are projected onto a measure of slope:

$$Proj[xr_{t+\Delta}^{n-\Delta} | y_{n,t} - y_{\Delta,t}] = (1 - \phi_n)(y_{n,t} - y_{\Delta,t}) \tag{31}$$

where $xr_{t+\Delta}^{n-\Delta}$ denotes one-month excess returns on the $n - \Delta$ period bond, realized at time $t + \Delta$. Combining with the established empirical fact that the Campbell and Shiller (1991) loadings ϕ_n's are always below one (and mostly negative), (31) clearly reveals that higher slope must be followed by higher returns. It follows that the positive (1,2) entry of the

feedback matrix, which counter-factually implies that higher slope must be followed by lower returns, is likely the key weak point of the $A_1(N)$ models. Moreover, the same weakness also applies to the $A_2(N)$ models as the (1,2) entries for these models are also similarly positive.

To examine whether the no-arbitrage restrictions are indeed forcing the physical feedback matrix of the stochastic volatility models to admit these counter-factual values, we estimate the $F_M(N)$ models established in Section 3. Recall that these are the counter-parts to the $A_M(N)$ models with the no-arbitrage restrictions, and thus the "first moments" restrictions through $A(\mathbb{P})$, completely relaxed. We use the same sample period (1973 through 2007) and the same set of yields in estimation, thus the estimated $F_M(N)$ and $A_M(N)$ models are directly comparable. Examining the reported values of the K_1 matrix reported in the last two panels of Table 3, for all four $F_M(N)$ models (M=1,2 and N=3,4) the (1,2) entry of the feedback matrix is negative.

Although a negative (1,2) entry should now allow slope to forecast level with the right sign, the key question is whether the $F_M(N)$ models, without no arbitrage restrictions, can produce loadings ϕ_n's that match up with the Campbell and Shiller (1991) regression (31) in the data. The answer to this question is a definite yes! Examining the pattern of the loadings ϕ_n implied by the $A_1(3)$ and $A_2(3)$ models in Figure 1, we find the well-known result of Dai and Singleton (2002) in which these stochastic volatility models have a long way to go in matching the empirical Campbell and Shiller (1991) regression coefficients. Nevertheless, once the no arbitrage restrictions are dropped and the the $A_1(3)$ and $A_2(3)$ models turn into the corresponding $F_1(3)$ and $F_2(3)$ models, the model-implied ϕ_n's now become extremely close to their empirical counter-parts, arguably as close as those loadings implied by the $A_0(3)$ model. A graph (not reported) for four factor models shows very similar results.

In short, Figure 1 constitutes convincing evidence that the no arbitrage restrictions, and in particular needed to match $M_1(\mathbb{Q})$, seem directly behind the failure of the $A_M(N)$ models for $M > 0$ in explaining the deviations from the EH. In stark contrast, the admissibility restriction $A(\mathbb{P})$ under the physical measure – present in both the $A_M(N)$ as well as $F_M(N)$ models – appears largely inconsequential for a model's ability in matching the deviations from the EH.

6.2 Why does imposing no-arbitrage lead to slope predicting level with a positive sign?

Whereas it seems clear the presence of no-arbitrage forces slope to predict level with a positive sign, thereby impairing no arbitrage stochastic volatility models' ability to match the empirical Campbell and Shiller (1991) regression coefficients, the exact mechanism is not obvious. To shed light on this, and with an emphasis on intuition, we focus on the $A_1(N)$ models and present two heuristic results. First, we show that as long as the risk-neutral dynamics of the non-volatility factors are not too close to being explosive, the loadings of the volatility factor on the level and slope factors ($\beta(1)$ and $\beta(2)$) will have the same sign. Second, we show that, given the tendency of the volatility factor to be relatively more persistent than the slope factor, the sign constraint on $\beta(1)$ and $\beta(2)$ necessarily causes slope to predict level

with a positive sign.

To see the former result in the most simplified manner, let's focus on the two-factor model $A_1(2)$ and think of the level and slope factors simply as the one-year yield, and the spread between the ten-year and one-year yield, respectively. We can show in Appendix A that the volatility loadings (with the first entry normalized to one) can be written as:

$$\beta = (1, W_1 B_X^{\mathbb{Q}} (W_2 B_X^{\mathbb{Q}})^{-1}),$$

where $W_1 = (1, 0)$, $W_2 = (-1, 1)$. Furthermore, standard bond pricing calculations reveal that $B_X^{\mathbb{Q}} = \Delta \left(\frac{1-e^{\lambda_X^{\mathbb{Q}}}}{(1-e^{\lambda_X^{\mathbb{Q}}\Delta})}, \frac{1-e^{10\lambda_X^{\mathbb{Q}}}}{10(1-e^{\lambda_X^{\mathbb{Q}}\Delta})} \right)'$ where $\lambda_X^{\mathbb{Q}}$ denotes the risk-neutral eigenvalue corresponding to the non-volatility factor as in (14). A few algebraic steps show that :

$$\beta(2) = W_1 B_X^{\mathbb{Q}} (W_2 B_X^{\mathbb{Q}})^{-1} = \frac{1 - e^{\lambda_X^{\mathbb{Q}}}}{1 - e^{\lambda_X^{\mathbb{Q}}} - \frac{1-e^{10\lambda_X^{\mathbb{Q}}}}{10}}. \tag{32}$$

Clearly, as long as $\lambda_X^{\mathbb{Q}} \leq 0$, or equivalently the non-volatility factor is stationary, both the numerator and the denominator of the right hand side of (32) are positive. Therefore we can make the following statement for the $A_1(2)$ model:

1. *As long as the non-volatility factor is $\mathbb{Q}$-stationary, the loading of the volatility factor on slope will always be of the same sign as the loading on level.*

2. *To the extent that the loading of volatility on level is generally positive, it implies that the loading on slope is also positive.*

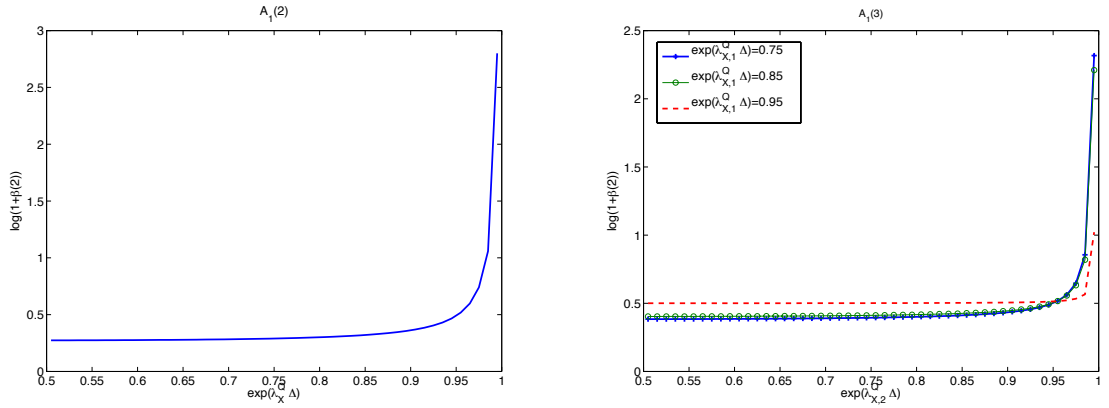
Similar results hold up for more general loadings of the level and slope factors and for the $A_1(3)$ model. Adopting the loadings W that correspond to the lower order yield PCs, we roam over the possible values of $\lambda_X^{\mathbb{Q}} \leq 0$ for both cases $N = 2$ and $N = 3$ and plot in Figure 5 the corresponding values of $1 + log(\beta(2))$ (again with $\beta(1)$ normalized to one). Note that this transformation, chosen for better scaling of the graphs, is positive if and only if $\beta(2)$ is positive. As can be seen clearly from the graphs, $\beta(2)$ is always positive, implying that both level and slope will load with the same sign in the volatility instrument of both the $A_1(2)$ and $A_1(3)$ models.

Turning to the second result, let's start by noting that the normalized volatility factor of the $A_1(2)$ model can be written as:

$$V_t = \beta \cdot \mathcal{P}_t = \mathcal{L}_t + \beta(2)\mathcal{S}_t \tag{33}$$

where $\mathcal{L}$ is the level, $\mathcal{S}$ is the slope, and $\beta(2)$ is positive. Now the admissibility restriction under the physical measure requires that only V_t can forecast $V_{t+\Delta}$, or equivalently:

$$E_t[V_{t+\Delta}] = constant + \rho_V V_t. \tag{34}$$



(a) $A_1(2)$ (b) $A_1(3)$

Figure 5: $log(1+\beta(2))$ for various values of $\lambda_X^{\mathbb{Q}}$.

Substitute (33) into (34) and evaluate the one month forecast of the level factor, we obtain:

$$E_t[\mathcal{L}_{t+\Delta}] = constant + \rho_V \mathcal{L}_t + \beta(2)(\rho_V \mathcal{S}_t - E_t[\mathcal{S}_{t+\Delta}]). \tag{35}$$

Assuming slope forecasts future slope with a coefficient of ρ_S,[21] then it follows from (35) that slope forecasts future level with a coefficient of

$$\beta(2)(\rho_V - \rho_S).$$

Due to the sign restriction $\beta(2) > 0$, established above, the sign with which slope forecasts level is dependent on the difference between ρ_V, the persistence of the volatility factor, and ρ_S. Empirically, due to well known volatility level effect, the volatility factor is typically quite persistent. In contrast, the slope factor operates at a relatively higher frequency. This suggests that ρ_S, which is closely related to the persistence of the slope factor, is likely much smaller than ρ_V, thus requiring slope to forecast level with a positive coefficient.

7 Risk price specification and identification of the volatility factor(s)

Within the class of $\mathbb{Q}$-affine term structure models, many different specifications for the market prices of risks have been proposed. Starting with the completely affine setup formalized by Dai and Singleton (2000), we have seen more flexible affine forms such as Duffee (2002), Cheridito, Filipovic, and Kimmel (2007), as well as non-affine forms such as Duarte (2004). Depending on the risk-price specifications, the physical dynamics can be affine or non-affine.

[21] To be more precise, $E_t[\mathcal{S}_{t+1}] = constant + \rho_S \mathcal{S}_t + \rho_{SL} \mathcal{L}_t$ but we focus only on the $\mathcal{S}_t$ term.

$A_0(3)$			$Duarte_1(3)$			$Diag_1(3)$		
0.997	0.028	0.025	0.998	0.028	0.024	0.998	0.028	0.024
-0.003	0.954	-0.098	-0.002	0.955	-0.097	-0.002	0.955	-0.098
-0.005	-0.002	0.928	-0.005	-0.000	0.931	-0.005	-0.000	0.930

Table 4: $K^{\mathbb{Q}}_{1,\Delta}$ Estimates

Nonetheless, a common thread through all of these different modelling choices is the fact that the risk-neutral dynamics of the underlying states remain affine and, importantly, free of artificial constraints beyond those that guarantee admissibility.

Recall from Section 5 that our regression-based estimates of the risk-neutral feedback matrix $K^{\mathbb{Q}}_{1,\Delta}$, which are completely independent of any physical dynamics, are quite close to estimates implied by the $A_M(N)$ models. It is therefore very likely that the risk neutral feedback matrix $K^{\mathbb{Q}}_{1,\Delta}$ will be very strongly identified regardless of their risk price specifications. Moreover, specializing to models with one volatility factor, due to $A(\mathbb{Q})$, the strong identification of $K^{\mathbb{Q}}_{1,\Delta}$ translates into a virtually discrete choice of the volatility instruments from the N left eigenvectors of $K^{\mathbb{Q}}_{1,\Delta}$. As is seen earlier, given the N left eigenvectors, the choice of which volatility instrument seems to rest on the matching of M_2 and much less on the functional form of risk prices. We therefore conjecture that the volatility factor implied by these models is likely highly similar across different risk price specifications.

To confirm our conjecture, we use the same data used earlier to estimate two term structure models with one volatility factor for $N = 3$ with different risk price specifications. The first adopts the non-affine approach of Duarte (2004) to include a square-root term in the risk price of the volatility factor. The second restricts the conditional feedback matrix corresponding to the PC yields portfolios to be diagonal under the physical measures. Similar diagonal restrictions have been considered by Joslin, Le, and Singleton (2012), among others. We refer to these models as $Duarte_1(3)$ and $Diag_1(3)$, respectively.

As is evident from Table 4, the risk neutral feedback matrices implied by $Duarte_1(3)$ and $Diag_1(3)$ are virtually identical and are extremely close to the $K^{\mathbb{Q}}_{1,\Delta}$ matrix implied by the constant volatility model $A_0(3)$ (which is shown earlier to be very close to that implied by the $A_1(3)$ model). Comparing the volatility factors implied by $Duarte_1(3)$ and $Diag_1(3)$ in Figure 6 clearly reveals that these volatility factors are virtually indistinguishable. Very similar results (not reported) are obtained for four factor models and for a shortened sample period that excludes the Fed regime. In short, we find strong evidence that altering the price of risk specification is relatively inconsequential for the identification of volatility.

8 Extensions

In this section, we show how our results extend to the case of multiple volatility factors and unspanned stochastic volatility. As before, this tension arises because of the relationship

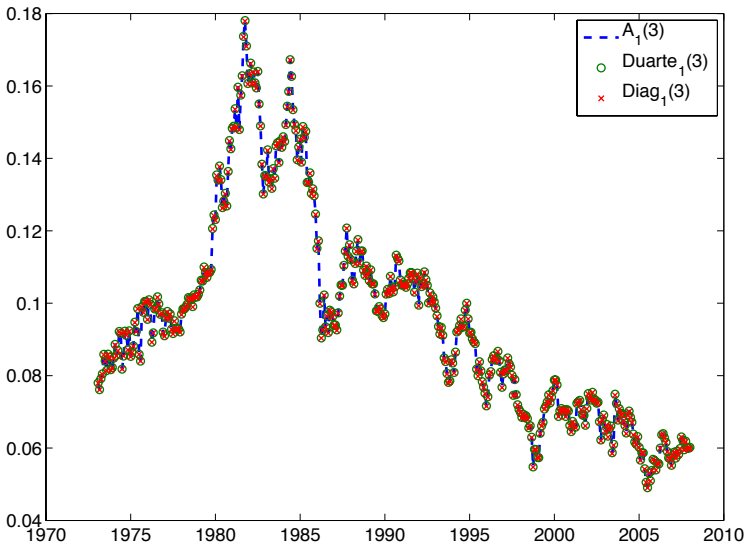


Figure 6: Volatility factor ($\beta \cdot \mathcal{P}_t$ where $\beta(1)$ is scaled to one) implied by the $A_1(3)$, $Duarte_1(3)$, and $Diag_1(3)$ models.

between the left eigenspaces of the feedback matrices under $\mathbb{P}$ and $\mathbb{Q}$. Additionally, new conditions arise because of requirements of positive feedback amongst CIR factors. We also show how our results extend to the case of unspanned stochastic volatility.

8.1 Multiple volatility factors

In an $A_M(N)$ model, as M rises relative to N, the restriction that the univariate volatility factor is an autonomous process weakens to the requirement that the M-dimensional volatility process is autonomous. This allows for richer feedback among the factors. While the conditions are weaker in this sense, new conditions arise where now feedback amongst the volatility factors must be positive. Additionally, such factors are now required to be independent. Although the proof of the results that we state below are generally direct, we omit them due to their tedious nature.

An $A_M(N)$ model has a latent factor representation as

$$dZ_t = (K_0 + K_1 Z_t)\, dt + + \sqrt{\Sigma_0 + \Sigma_{1,1} Z_{t1} + \ldots + \Sigma_{1M} Z_{t,M}}\, dB_t \tag{36}$$

with $r_t = \rho_0 + \rho_1 \cdot X_t$ and the admissibility requirements:

$$\begin{aligned} K_{0i} &\geq \frac{1}{2}\Sigma_{1,i,ii}, \quad i \leq M, && (37)\\ K_{1,ij} &\geq 0, \quad i,j \leq M, i \neq j, && (38)\\ K_{1,ij} &= 0, \quad i \leq M < j, && (39)\\ \Sigma_{0,ii} &= 0, \quad i \leq M, && (40)\\ \Sigma_{1,i,jj} &= 0, \quad i,j \leq M, i \neq j, && (41) \end{aligned}$$

and $\Sigma_0, \Sigma_1, \ldots, \Sigma_M$ are positive semi-definite symmetric matrices. Similar relations apply for the risk-neutral dynamics. As before we have the relation

$$y_t = A_Z^{\mathbb{Q}} + B_Z^{\mathbb{Q}} Z_t, \tag{42}$$

where $A_Z^{\mathbb{Q}}$ and $B_Z^{\mathbb{Q}}$ are dependent on the underlying parameters.

When $M > 1$ there will be multiple volatility instruments given by a set of vectors $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_M\}$. As before, we will require that under both the historical and risk-neutral measures there are CIR-type factors which do not have any feedback from the conditionally Gaussian factors. In the $A_1(N)$ case, this required that β_1 is a left eigenvector of K_1 and $K_1^{\mathbb{Q}}$. For the general case of $M \geq 1$, we require that there exists M left eigenvectors of K_1, $\{e_1, \ldots, e_M\}$, and M left eigenvectors of $K_1^{\mathbb{Q}}$, $\{f_1, \ldots, f_M\}$ so that the span of $\mathcal{B}$ is the same of the span of both $\{e_1', \ldots, e_M'\}$ and $\{f_1', \ldots, f_M'\}$.[22] This will give a direct tension between first moments.

The conditions for no-feedback between conditionally Gaussian and CIR factors required by (39) imply eigenvector restrictions that give rise to a tension between $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$. The restriction of positive feedback among the CIR factors given by (38) also implies a tension between $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$. To illustrate these tensions, let us consider the case of an $A_2(2)$ model. In this case, since $M = N$, the eigenvector conditions do not bind.

Suppose that $K_1^{\mathbb{Q}}$ has real eigenvalues $\{\lambda_1, \lambda_2\}$. with corresponding eigenvectors $\{u_1, u_2\}$. Then $UK_1^{\mathbb{Q}}U^{-1} = \text{diag}(\lambda_1, \lambda_2)$ where $U = [u_1', u_2']$. Straightforward but tedious computations imply that the positive feedback conditions in (38) requires that $\Gamma = UK_1U^{-1}$ must satisfy one of:

1. $\Gamma_{12}\Gamma_{21} \geq 0$, or
2. $\Gamma_{21} < 0 < \Gamma_{12}$ and $\Gamma_{22} - \Gamma_{11} > 2\sqrt{-\Gamma_{12}\Gamma_{21}}$, or
3. $\Gamma_{12} < 0 < \Gamma_{21}$ and $\Gamma_{11} - \Gamma_{22} > 2\sqrt{-\Gamma_{12}\Gamma_{21}}$.

Note that by our previous logic, $K_1^{\mathbb{Q}}$ is likely to be strongly identified by the cross-section of yields and largely invariant to the choice of M. Therefore U should be estimated very precisely. If one considers U to be fixed (i.e. estimated without error), then the conditions

[22] Alternatively, we can characterize the relation that the exists a single fixed matrix U so that for each i, both (i) $K_1'(U\beta_i)$ is in the span of $\mathcal{B}$ and (ii) $(K_1^{\mathbb{Q}})'(U\beta_i)$ is in the span of $\mathcal{B}$.

above translate into quadratic constraints on K_1. This is the new tension between $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$ not present in $A_1(N)$ models.

In addition, the tensions with M_2 become more pronounced. To see this, suppose that $\{c_1 \cdot \mathcal{P}_t, c_2 \cdot \mathcal{P}_t\}$ have uncorrelated innovations. That is, the covariance matrix of innovations to $C\mathcal{P}_t$, where $C = [c_1', c_2']'$, is diagonal. Then it must be that CK_1C^{-1} and $CK_1^{\mathbb{Q}}C^{-1}$ has non-negative off-diagonal elements. Again taking the extreme example that $K_1^{\mathbb{Q}}$ is estimated without any error, this will impose a restriction on which possible linear combinations of $\mathcal{P}_t$ are uncorrelated.

8.2 Unspanned stochastic volatility

With some modifications, our results also apply in models with unspanned stochastic volatility as in Collin-Dufresne and Goldstein (2002) and Bikbov and Chernov (2009), among others. We now discuss these results for the case of one or more stochastic volatility factors as well as for the cases with purely unspanned stochastic volatility or a mixture of spanned and unspanned stochastic volatility. Similar results would apply in cases with unspanned risk such as inflation risk (see, e.g. Chernov and Mueller (2009)), provided additional data is available to price this risk such as inflation swaps.

In the case of an $A_1(N)$ model with unspanned stochastic volatility, the tension implies that volatility is an autonomous process under $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$. The clean separation of yields and volatility will imply that our results are informative only about volatility and will require that volatility is an autonomous process, which is well understood in the literature. For example, Almeida, Graveline, and Joslin (2011) find correlations between yield curve factors and volatility measures as high as 84% while Jacobs and Karoui (2009) find R^2 as high as 75%.

When $M > 1$, several possibilities arise. Joslin (2013a) develops an $A_2(N)$ model where there are both spanned and unspanned stochastic volatility. Such a model would be consistent with the moderate R^2 (but less that 100%) that are available from projecting volatility/variance measures onto the yield curve. Our existing results will apply directly to the spanned component of volatility. For example, the possible choices for the spanned instrument for volatility will be determined by $K_1^{\mathbb{Q}}$. Moreover, spanned and unspanned volatility factors will follow an $A_2(2)$ process. The previous discussion of $A_2(2)$ models will now apply. In this case, $M_1(\mathbb{P})$ and $M_1(\mathbb{Q})$ will be determined by actual and risk-neutral forecasts of volatility. Risk-neutral forecasts of volatility will now be identified by a cross-section of option prices instead of bond prices. For example, if $V_t = (V_t^{span}, V_t^{unspan})$ and CV_t has uncorrelated innovations, then our results in Section 8.1 show that the risk-neutral and actual forecasts will have to satisfy a number of constraints.

Another alternative is to have a model with multiple unspanned stochastic volatility factors. Joslin (2013a) and Trolle and Schwartz (2009) show how to construct such models. In this case, our results regarding $A_2(2)$ models (and more generally $A_N(N)$) will apply to the volatility factors. Thus, again, there will be tensions as we have outlined between risk-neutral and actual forecasts of volatility as well as the volatility of volatility. Moreover, our logic would generally apply to multiple unspanned volatility factors where the risk-neutral

dynamics could be identified by options as in Carr, Gabaix, and Wu (2009).

9 Conclusion

In the context of no arbitrage affine term structure model with stochastic volatility, we document a strong tension between the first moments of bond yields under the time series and risk neutral measures. We show that, beyond other types of tensions documented so far in the existing literature, this tension is key in understanding important economic implications of no-arbitrage affine models with stochastic volatility. In particular, this tension underlies the well-known failure of the $A_M(N)$ class of models in explaining the deviations from the EH in bond data.

Our primary results are driven by the fact that an affine drift requires a number of constraints in order to assure that volatility stays positive. A number of alternative models could be considered. First, one could consider a model with unspanned or nearly unspanned volatility. This, however, can only partially counteract our results in the sense that the projection of volatility onto yields must still mathematically be a positive process. So several of our insights maintain. Another possible model to consider is a model with non-linear drift. That is, we can suppose that there is a latent state variable Z_t with the drift of Z_t linear in Z_t without any constraints provided that volatility (or its instrument) is far from the boundary. Near the zero boundary, the drift of the volatility may be non-linear in such a way as to maintain positivity. Provided that the probability of entering this non-linear region is small (under $\mathbb{Q}$), similar pricing equation will be obtained as in the standard affine setting.

A Dependence of Volatility Loadings β and the Eigenvalues under $\mathbb{Q}$

Our arguments in Section 3.3 rest on the approximation that convexity effects are negligible. Although we confirm empirically that this approximation holds up in the data, we now make our arguments more precise by showing that the volatility instrument β is in fact completely determined by the $(N-1)$ eigenvalues given in $\lambda_X^{\mathbb{Q}}$. This can be seen as follows. Let $\mathcal{P}_t^{(1)} = W_1 y_t$ ($\mathcal{P}_t^{(2)} = W_2 y_t$) denote the first entry (entries two to N) of $\mathcal{P}_t$ where W_1 (W_2) refers to the first row (rows 2 to N) of W. Let $B_V^{\mathbb{Q}}$ and $B_X^{\mathbb{Q}}$ denote the yield loadings on V_t and X_t, respectively. Thus, $B_V^{\mathbb{Q}}$ corresponds to the first column, and $B_Z^{\mathbb{Q}}$ the remaining columns of $B_Z^{\mathbb{Q}}$. Notably, due to the block structure of the feedback matrix in (14) and the fact that the non-volatility factor X_t does not give rise to Jensen effects, it can be shown that $B_X^{\mathbb{Q}}$ only depends on $\lambda_X^{\mathbb{Q}}$. Then we have,

$$\begin{aligned}\mathcal{P}_t^{(1)} &= \text{constant} + W_1 B_V^{\mathbb{Q}} V_t + W_1 B_X^{\mathbb{Q}} X_t \\ \mathcal{P}_t^{(2)} &= \text{constant} + W_2 B_V^{\mathbb{Q}} V_t + W_2 B_X^{\mathbb{Q}} X_t.\end{aligned}$$

This gives two equations and two unknowns, so we can subtract $W_1 B_X^{\mathbb{Q}} (W_2 B_X^{\mathbb{Q}})^{-1}$ times the second equation from the first equation to eliminate X_t and obtain

$$\mathcal{P}_t^{(1)} - W_1 B_X^{\mathbb{Q}} (W_2 B_X^{\mathbb{Q}})^{-1} \mathcal{P}_t^{(2)} = \text{constant} + cV_t,$$

where c is a constant. This shows directly that no arbitrage imposes the restriction (up to scaling):

$$\beta = (1, -W_1 B_X^{\mathbb{Q}} (W_2 B_X^{\mathbb{Q}})^{-1}). \tag{43}$$

We see that the volatility instrument is in fact determined entirely by $\lambda_X^{\mathbb{Q}}$. Coupled with our reasoning earlier that $\lambda_X^{\mathbb{Q}}$ should be strongly pinned down in the data, it is clear that the volatility instruments are heavily affected by the no arbitrage restrictions. Equation (43) also makes clear the nature of the (close) relationship between the volatility instrument and yield loadings discussed earlier.

B A Canonical Form for Discrete-Time Term Structure with Stochastic Volatility

In this section, drawing on the construction in Le, Singleton, and Dai (2010) (LSD), we lay out canonical forms for discrete-time affine term structure models with stochastic volatility. As is shown by LSD, for monthly data, these provide very good approximation to the continuous time $A_M(N)$ models in the main text.

We start by assuming that the economy is fully characterized by the N-variate state vector $Z_t = (V_t', X_t')'$ where V_t is a strictly positive M-variate volatility process and X_t is conditionally Gaussian. The time interval is Δ.

B.1 Risk-neutral dynamics and bond pricing

Under Q, our states follow:

$$V_{t+\Delta}|V_t \sim CAR(\rho^{\mathbb{Q}}, c^{\mathbb{Q}}, \nu^{\mathbb{Q}}), \tag{44}$$

$$X_{t+\Delta} \sim N(K^{\mathbb{Q}}_{1V,\Delta}V_t + K^{\mathbb{Q}}_{1X,\Delta}X_t, \Sigma_{0,\Delta} + \sum_{i=1}^{M}\Sigma_{i,\Delta}V_{i,t}), \text{ independent of } V_{t+\Delta} \tag{45}$$

$$r_t = r_\infty + \rho_V'V_t + \iota'X_t. \tag{46}$$

CAR denotes a compound autoregressive gamma process. See LSD for more details. Each CAR process is fully characterized by three non-negative parameters: $\rho^{\mathbb{Q}}$ $(M \times M)$, $c^{\mathbb{Q}}$ $(M \times 1)$, and $\nu^{\mathbb{Q}}$ $(M \times 1)$. The Laplace transform for a CAR variable:

$$E_t[e^{uZ_{t+1}}] = e^{a(u)+b(u)Z_t} \text{ where } a(u) = -\sum \nu_i^{\mathbb{Q}} log(1-u_ic_i^{\mathbb{Q}}), \; b(u) = \sum \rho_i^{\mathbb{Q}}\frac{u_i}{1-u_ic_i^{\mathbb{Q}}},$$

where the subscript i indexes the i^{th} element for vectors and the i^{th} row for matrices.

From the Laplace transform, standard bond pricing calculations show that bond prices for all maturities are exponentially affine. Denoting by $P_{n,t}$ the price of a zero coupon bond with n periods ($n\Delta$ years) until maturity, we can show that $logP_{n,t} = -A_n - B_{V,n}V_t - B_{X,n}X_t$ with loadings given by:

$$B_{X,n} = \iota' + B_{X,n-1}K^{\mathbb{Q}}_{1X}, \tag{47}$$

$$B_{V,n,i} = \rho_{V,i} + \sum_{k=1}^{M}\left(\rho^{\mathbb{Q}}(k,i)\frac{B_{V,n-1,k}}{1+B_{V,n-1,k}c_k^{\mathbb{Q}}} + B_{X,n-1}K^{\mathbb{Q}}_{1V}(:,i) - \frac{1}{2}B_{X,n-1}\Sigma_{i,X}B'_{X,n-1}\right), \tag{48}$$

$$A_n = r_\infty + A_{n-1} + \sum \nu_i^{\mathbb{Q}} log(1+B_{V,n-1,i}c_i^{\mathbb{Q}}) - \frac{1}{2}B_{X,n-1}\Sigma_{0X}B'_{X,n-1}, \tag{49}$$

starting from: $A_0 = B_{V,0} = B_{X,0} \equiv 0$.

B.2 Physical dynamics

Under P, the state variables follow:

$$V_{t+\Delta}|V_t \sim \text{bivariate } CAR(\rho, c, \nu), \tag{50}$$

$$X_{t+\Delta} \sim N(K_{0,\Delta} + K_{1V,\Delta}V_t + K_{1X,\Delta}X_t, \Sigma_{0,\Delta} + \sum_{i=1}^{M}\Sigma_{i,\Delta}V_{i,t}), \text{ independent of } V_{t+\Delta} \tag{51}$$

Non-attainment under $\mathbb{P}$ requires the Feller condition: $\nu \geq 1$.

B.3 The continuous time limit

The conditional mean $E_t[V_{t+1}]$ and conditional covariance matrix $V_t[Z_{t+1}]$ implied by the Laplace transform of the CAR process are

$$E_t^{\mathbb{P}}[V_{t+1}](i) = \nu_i c_i + \rho_i V_t,\ V_t^{\mathbb{P}}[V_{t+1}](i,i) = \nu_i c_i^2 + 2c_i\rho_i Z_t, \tag{52}$$

and the off-diagonal elements of $V_t^{\mathbb{P}}[V_{t+1}]$ are all zero (correlation occurs only through the feedback matrix).

That this process converges to the multi-factor CIR process can be seen by letting $\rho = I_{M\times M} - \kappa\Delta t$, $c_i = \frac{\sigma_i^2}{2}\Delta t$, and $\nu_i = \frac{2(\kappa\theta)_i}{\sigma_i^2}$, where κ is a $M \times M$ matrix and θ is a $M \times 1$ vector. In the limit as $\Delta t \to 0$, the V_t converges to:

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{diag(V_t)}dB_t,$$

where σ is a $N \times N$ diagonal matrix with i^{th} diagonal element given by σ_i.

For the conditionally Gaussian variables, it is straightforward to see that if we let $K_{0,\Delta} = K_0\Delta t$, $K_{1V,\Delta} = K_{1V}\Delta t$, $K_{1X,V} = I - K_{1X}\Delta t$, and $\Sigma_{i,\Delta} = \Sigma_i\Delta t$, in the time limit, the X_t process converges to:

$$dX_t = (K_0 + K_{1V}V_t + K_{1X}X_t)dt + \sqrt{\Sigma_0 + \sum_{i=1}^{M}\Sigma_i V_{i,t}}dB_t. \tag{53}$$

B.4 Technical conditions

We now discuss two technical issues related to this parameterization. First, consider the market prices of variance risk:

$$Var_t^{\mathbb{P}}[V_{t+1}]^{-1}(E_t^{\mathbb{P}}[V_{t+1}] - E_t^{\mathbb{Q}}[V_{t+1}]).$$

As discussed by Cheridito, Filipovic, and Kimmel (2007), when $Var_t^{\mathbb{P}}[V_{t+1}]$ approaches zero, there is the issue of exploding market prices of risks unless the intercept terms of $E_t^{\mathbb{P}}[V_{t+1}]$ and $E_t^{\mathbb{Q}}[V_{t+1}]$ are the same (hence the numerator too approaches zero at the same rate as the denominator). Nevertheless, in our discrete time setup, as long as ν and c are strictly positive, $Var_t^{\mathbb{P}}[V_{t+1}]$ is bounded strictly away from zero. As a result, we don't have to directly deal with this issue. If one wishes to avoid this issue even in the continuous time limit, then a sufficient restriction on the parameters is:

$$vc = v^{\mathbb{Q}}c^{\mathbb{Q}}.$$

Finally, the scale parameters (c and $c^{\mathbb{Q}}$) in principle can be any pair of positive numbers in our discrete time setup. Nevertheless, the diffusion invariance property of the CIR process requires that these two parameters have the same continuous time limit ($\frac{1}{2}\sigma^2 dt$). To be consistent with diffusion invariance of V_t in the continuous time limit, then a sufficient restriction on the parameters is:

$$c = c^{\mathbb{Q}}.$$

C Estimation

For estimation, we use the monthly unsmoothed Fama Bliss zero yields with eleven maturities: 6month, one- out to ten-year. We start our sample in January 1973, due to the sparseness of longer maturity yields prior to this period, and end in December 2007 to ensure our results are not influenced by the financial crisis.

Using the canonical form laid out in Appendix B and assuming that the first N PCs of bond yields are priced perfectly and the remaining PCs are priced with iid errors and one common variance, we compute the model implied one-month ahead conditional means and variances and implement estimation using QMLE.

In the main text, we note that the one-month ahead conditional mean of the yields portfolios $\mathcal{P}_t$ take an affine form. The same also holds for the one-month ahead conditional variance. That is:

$$E_t^{\mathbb{P}}[\mathcal{P}_{t+\Delta}] = K_{0,\Delta} + K_{1,\Delta}\mathcal{P}_t, \tag{54}$$

$$E_t^{\mathbb{Q}}[\mathcal{P}_{t+\Delta}] = K_{0,\Delta}^{\mathbb{Q}} + K_{1,\Delta}^{\mathbb{Q}}\mathcal{P}_t, \tag{55}$$

$$Var_t^{\mathbb{P}}[\mathcal{P}_{t+\Delta}] = \Sigma_{0,\mathcal{P},\Delta} + \sum_{i=1}^{M} \Sigma_{i,\mathcal{P},\Delta} V_{i,t}, \tag{56}$$

where $V_t = \alpha + \beta'\mathcal{P}_t$. Thus one way to fully characterize each of the $A_M(N)$ model is through the set of parameters $\Theta_\Delta = (K_{0,\Delta}, K_{1,\Delta}, K_{0,\Delta}^{\mathbb{Q}}, K_{1,\Delta}^{\mathbb{Q}}, \Sigma_{i,\mathcal{P},\Delta}, \alpha, \beta)$. In the main text, we have reported estimates of $K_{1,\Delta}$ and $K_{1,\Delta}^{\mathbb{Q}}$ for the $A_M(N)$ models. For completeness, we report here all the remaining estimates. Table 5 contains estimates of the intercept terms $K_{0,\Delta}$ and $K_{0,\Delta}^{\mathbb{Q}}$. Table 6 reports the estimates of the volatility loadings α and β. Table 7 and Table 8 report the Choleskey decomposition of the variance parameters $\Sigma_{i,\mathcal{P},\Delta}$ for 4-factor models and 3-factor models, respectively.

	$\mathbb{P}$			$\mathbb{Q}$		
	M=0	M=1	M=2	M=0	M=1	M=2
$A_M(4)$	0.04	0.05	0.09	0.01	0.01	0.01
	-0.03	0.01	0.10	-0.03	-0.04	-0.04
	-0.30	-0.24	-0.26	-0.04	-0.04	-0.03
	0.24	0.23	0.12	-0.07	-0.10	-0.09
$F_M(4)$	0.04	-0.02	-0.07			
	-0.03	0.02	0.04			
	-0.30	-0.25	-0.19			
	0.24	0.23	0.24			
$A_M(3)$	0.03	-0.03	0.02	0.01	0.01	0.01
	-0.05	-0.10	-0.05	-0.02	-0.04	-0.03
	-0.28	-0.21	-0.25	-0.05	-0.06	-0.06
$F_M(3)$	0.03	-0.04	-0.02			
	-0.05	0.01	0.04			
	-0.28	-0.25	-0.24			

Table 5: Estimates of $K_{0,\Delta}$ and $K^{\mathbb{Q}}_{0,\Delta}$

	$A_1(N)$		$A_2(N)$			$F_1(N)$		$F_2(N)$		
	α	β	α	β		α	β	α	β	
N=4	-7.20	1.94	-6.58	1.75	1.63	-5.43	2.00	-3.87	1.07	1.13
		1.32	5.80	1.28	-0.58		-0.64	-2.94	-0.18	-0.28
		-0.50		-0.47	1.40		-1.29		-0.57	-0.90
		-0.70		-0.74	0.72		-0.20		0.79	-0.84
N=3	-10.32	2.75	-10.41	2.74	1.23	-1.30	1.00	0.72	3.20	2.19
		1.79	21.87	1.82	0.65		-0.54	-2.66	3.90	-1.21
		-1.58		-1.60	2.99		-0.32		-1.32	-0.67

Table 6: Estimates of α and β

	$\Sigma_{0,\mathcal{P},\Delta}$				$\Sigma_{1,\mathcal{P},\Delta}$				$\Sigma_{2,\mathcal{P},\Delta}$			
$A_0(4)$	0.40											
	-0.11	0.40										
	0.11	0.07	0.40									
	0.01	-0.02	-0.02	0.62								
$A_1(4)$	0.08				0.12							
	-0.00	0.01			-0.00	0.13						
	0.20	-0.03	0.00		0.03	0.02	0.11					
	0.05	-0.02	0.00	0.00	0.01	-0.01	-0.01	0.16				
$A_2(4)$	0.03				0.08				0.07			
	-0.01	0.01			0.02	0.13			-0.03	0.01		
	0.03	-0.01	0.00		-0.05	0.04	0.08		0.08	-0.01	0.00	
	0.02	-0.02	0.00	0.00	-0.06	0.01	-0.13	0.07	0.06	-0.02	0.00	0.00
$F_0(4)$	0.40											
	-0.11	0.40										
	0.11	0.07	0.40									
	0.01	-0.02	-0.02	0.62								
$F_1(4)$	0.21				0.10							
	0.18	0.18			-0.06	0.07						
	0.23	-0.07	0.05		-0.00	-0.03	0.12					
	0.01	-0.09	0.02	0.00	0.02	0.02	-0.02	0.18				
$F_2(4)$	0.19				0.10				0.10			
	0.18	0.17			-0.06	0.02			-0.04	0.11		
	0.23	-0.05	0.04		-0.02	-0.09	0.03		0.02	-0.02	0.13	
	-0.05	-0.00	-0.01	0.04	0.13	0.09	0.11	0.00	-0.08	0.01	0.07	0.12

Table 7: Estimates of $\Sigma_{i,\mathcal{P},\Delta}$ (choleskey decomposition) for $A_M(4)$ and $F_M(4)$ models

	$\Sigma_{0,\mathcal{P},\Delta}$			$\Sigma_{1,\mathcal{P},\Delta}$			$\Sigma_{2,\mathcal{P},\Delta}$		
	0.41								
$A_0(3)$	-0.11	0.40							
	0.10	0.07	0.41						
	0.10			0.09					
$A_1(3)$	0.07	0.00		-0.02	0.10				
	0.25	-0.03	0.00	0.01	0.00	0.08			
	0.03			0.08			0.03		
$A_2(3)$	0.02	0.00		-0.03	0.09		0.02	0.00	
	0.07	-0.02	0.00	-0.02	-0.02	0.02	0.07	-0.01	0.00
	0.40								
$F_0(3)$	-0.11	0.40							
	0.11	0.07	0.41						
	0.19			0.14					
$F_1(3)$	0.23	0.11		-0.09	0.09				
	0.21	-0.17	0.06	0.01	-0.02	0.14			
	0.09			0.03			0.09		
$F_2(3)$	0.02	0.03		0.05	0.01		-0.07	0.03	
	0.26	-0.02	0.06	0.01	-0.02	0.00	0.02	0.08	0.04

Table 8: Estimates of $\Sigma_{i,\mathcal{P},\Delta}$ (choleskey decomposition) for $A_M(3)$ and $F_M(3)$ models

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